UCI Math Circle **September 14, 2013**

Prime numbers and Diophantine equations

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1. Warm-up

a. Given a 3 gallon jug and a 5 gallon jug (without any markings), is it possible to get exactly 1 gallon of water from a well? If so, how? If not, why not?

b. In how many ways can a debt of $69 be paid using only $5 and $2 bills?

2. Diophantine equations

For each of the following exercises, find as many integer solutions as you can.

a. 3□+5‣=1

b. 4□+6‣=10

c. 4□+6‣=7

Observations:
We notice that if a+b=c, with a and b even numbers, then c must be __________.

d. Are there any solutions to 5□+10‣=11?

Observations:

e. If 6□+12‣=k, what can be said about the number k?

f. If 30□+25‣=m, what can be said about the number m?

Make your own theorem:

If ? is ____________, then 30□+?‣=20 has no solutions.

3. Prime Numbers

Definition: A natural number larger than 1 is called prime if its only divisors are 1 and itself.

Examples of prime numbers: 2, 3, 5, 7, ...

a. Can you name the next 5 prime numbers?

b. What is the largest prime number you know?

c. How many prime numbers have been found?

d. How many prime numbers are there?

Try out the Sieve of Eratosthenes:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Unique Factorization Theorem:
Every positive integer other than 1 can be factored into prime factors in exactly one way (except possibly for the order of the factors).

Example: **140** $= 14 \cdot 10 = 2 \cdot 7 \cdot 2 \cdot 5 = 2^2 \cdot 5 \cdot 7$

4. Factor each of the following numbers into a product of their prime factors.
   a. 60

   b. 7280

   c. 107

   d. 48944

   e. 7900200

Theorem: There are infinitely many prime numbers.

Why?

Types of Primes:

A **Mersenne prime** is a prime number of the form 2^n-1. The Great Internet Mersenne Prime Search is a computer program that uses the computing power of thousands of volunteers to search for prime numbers. The largest known prime number is $2^{43,112,609}$-1. It has 12,978,189 digits and was found in January of 2013.

How many Mersenne primes do you know?

Is every number of the form 2^n-1 prime?

What if we require p to be prime, is 2^p-1 always prime?

If n>2 is even, then 2^n-1 is never prime. Why?

What about 2^n-1 for n a multiple of 3?