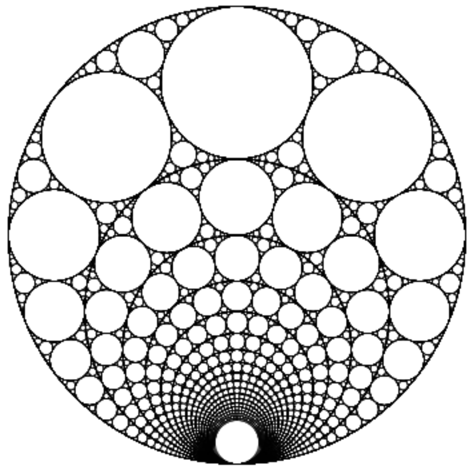
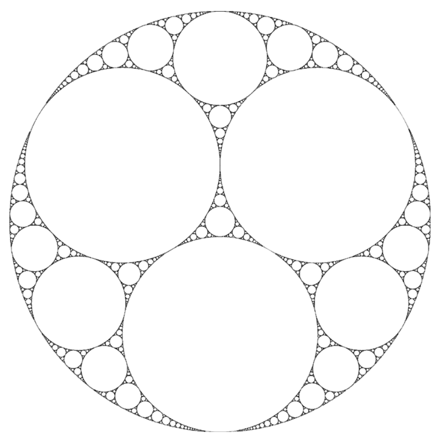
Fractals

A *fractal* is a shape that seem to have the same structure no matter how far you zoom in, like the figure below.



Sometimes it's only part of the shape that repeats. In the figure below, called an Apollonian Gasket, no part looks like the whole shape, but the parts near the edges still repeat when you zoom in.



Today you'll learn how to construct a few fractals:

- The Snowflake
- The Sierpinski Carpet
- The Sierpinski Triangle
- The Pythagoras Tree
- The Dragon Curve

After you make a few of those, try constructing some fractals of your own design!

There's more on the back. →

Challenge Problems

In order to solve some of the more difficult problems today, you'll need to know about the *geometric series*. In a geometric series, we add up a sequence of terms, each of which is a fixed multiple of the previous one. For example, if the ratio is $\frac{1}{2}$, then a geometric series looks like

$$1 + \frac{1}{2} + \left(\frac{1}{2} \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) + \ldots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$

The geometric series has the incredibly useful property that we have a good way of figuring out what the sum equals. Let's let r equal the common ratio (like $\frac{1}{2}$ above) and n be the number of terms we're adding up. Our series looks like

$$1 + r + r^2 + \ldots + r^{n-2} + r^{n-1}$$

If we multiply this by $1 - r$ we get something rather simple.

$$(1 - r)(1 + r + r^2 + \ldots + r^{n-2} + r^{n-1})$$

$$=$$

$$\begin{array}{ccccccccccccc} 1 & + & r & + & r^2 & + & \ldots & + & r^{n-2} & + & r^{n-1} & & \\ - & (& r & + & r^2 & + & \ldots & + & r^{n-2} & + & r^{n-1} & + & r^n \;) \end{array}$$

$$=$$

$$1 \qquad\qquad\qquad\qquad\qquad\qquad - \; r^n$$

Thus

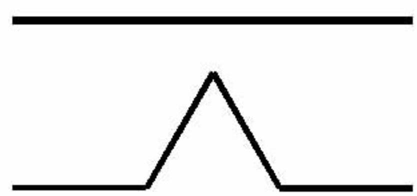
$$1 + r + r^2 + \ldots + r^{n-2} + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

If we're clever, we can use this formula to compute the areas and perimeters of some of the shapes we create.

Snowflakes

1. Let's learn how to make a fractal snowflake! First, a bit of practice to get the pattern.

    (a) Start by drawing a straight line segment. Divide it into thirds, and then replace the middle third with an equilateral triangle, like so:

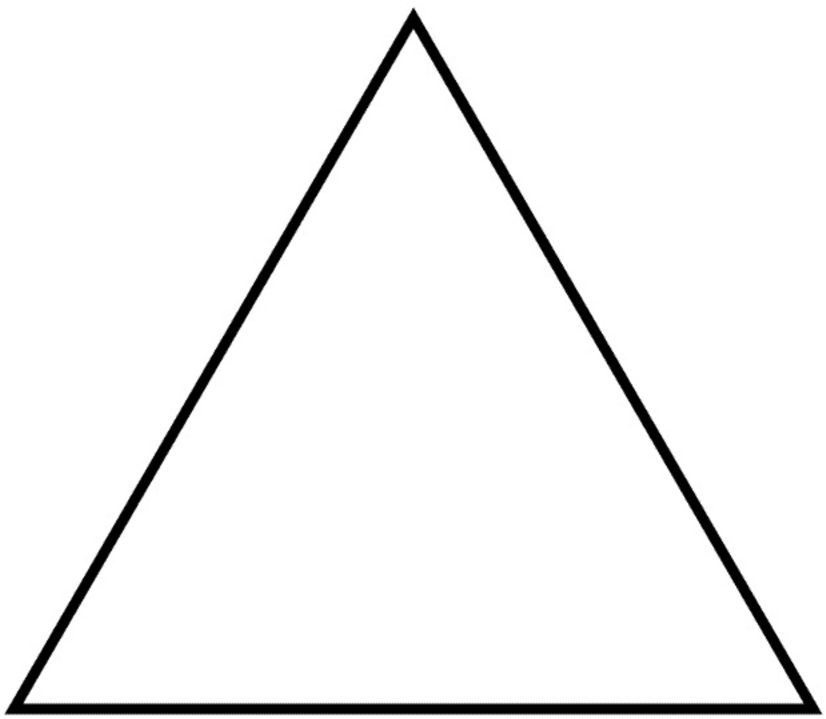
    Continue this process for each of the four segments of the new figure, replacing the middle third with an equilateral triangle. Try to see how many times you can repeat the process.

2. Once you've got the hang of it, draw an equilateral triangle (or find a piece of paper that already has one) and apply the rules to the sides of the triangle. To get a good idea of what the fractal snowflake looks like, repeat the process at least two more times.

3. Find the perimeter of the snowflake:

Figure	Perimeter
Original Triangle	
1 Iteration	
2 Iterations	
3 Iterations	
4 Iterations	
n Iterations	

4. Intriguing Question: Imagine you repeat the process infinitely many times. What happens to the perimeter? Is your (finite) snowflake still going to have a finite perimeter?



The Sierpinski Triangle

1. Making the Sierpinski triangle.

   (a) Start by drawing a large triangle. Any triangle will work, but larger is better.

   (b) Find the midpoints of each side.

   (c) Connect them with straight lines to form a new triangle.

   (d) Shade this triangle in. You should now have three smaller, unshaded triangles.

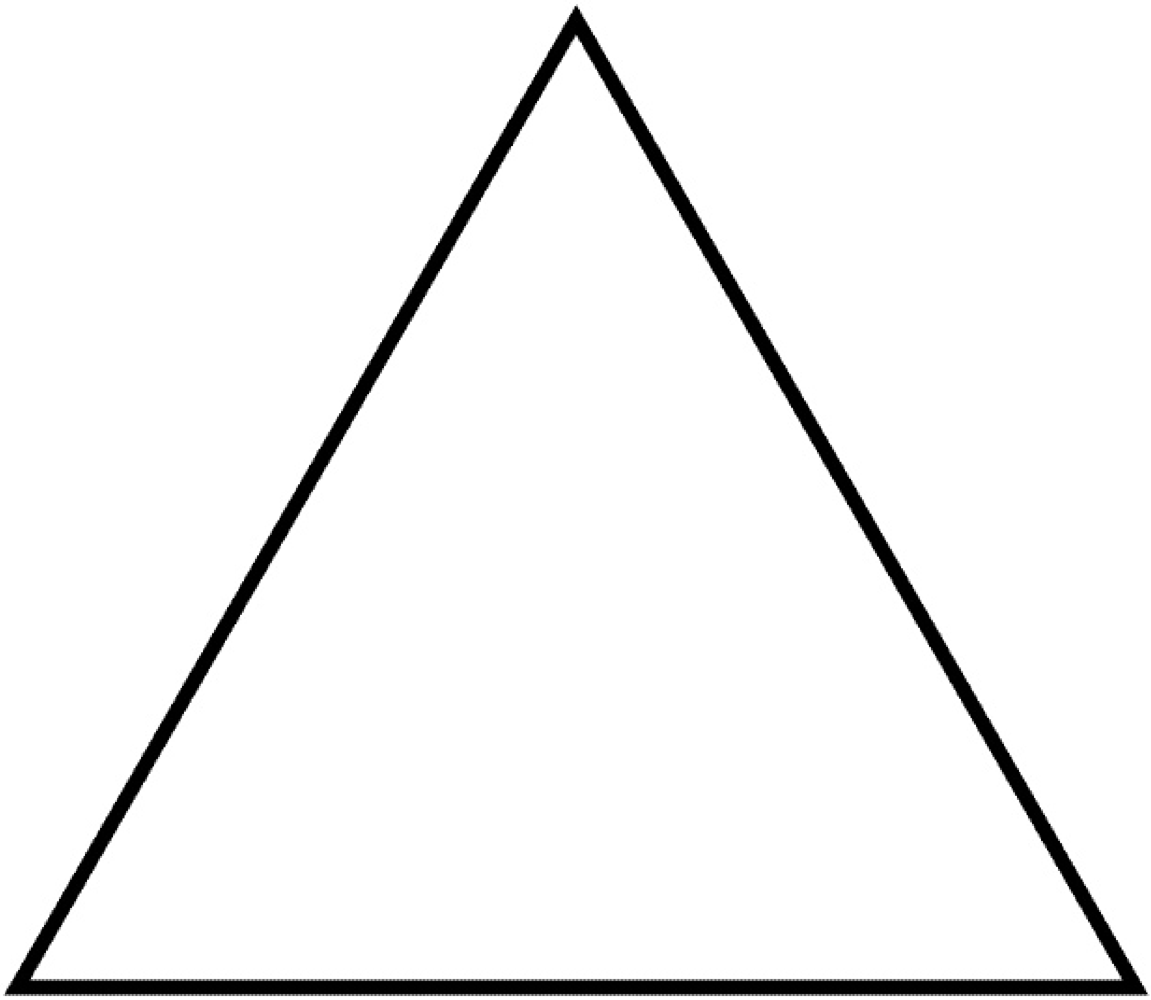
   (e) Repeat this process with the sides of each of the smaller triangles. How many unshaded triangles does this give you?

   (f) Repeat this process as many times as you'd like. The true Sierpinski triangle has the process repeated an infinite number of times!

2. Find the total perimeter of the unshaded triangles:

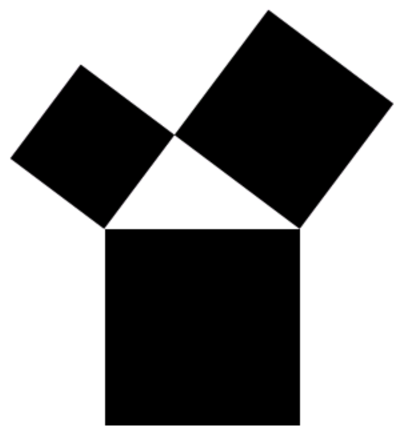
Figure	Triangle Perimeter	Total Perimeter
Original Triangle		
1 Iteration		
2 Iterations		
3 Iterations		
4 Iterations		
n Iterations		

3. If you could extend the process infinitely, what would you expect the total perimeter of the unshaded triangles to be?



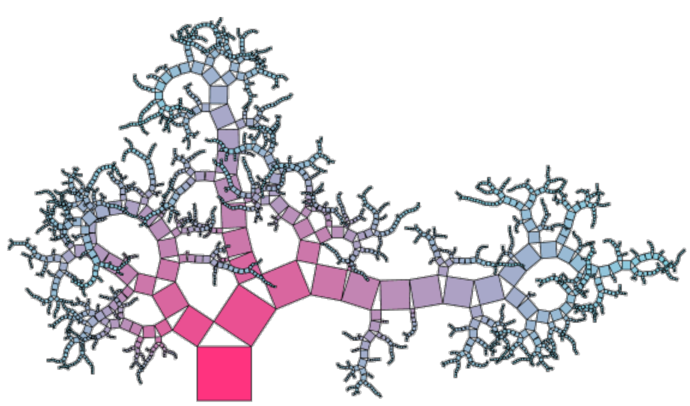
The Pythagoras Tree

1. Making the Pythagoras tree.
   (a) Start by drawing a medium square and shading it in.
   (b) Draw a right triangle on top of the square. It'll be easiest if the two new sides have the same length, but that's not required.
   (c) Draw and shade in squares on the two other sides of the triangle. (Have fun shading triangles and squares with different colors.)



   (d) Repeat this process with each of the new squares. Draw triangles with the same proportions as your first one. Why do you think this is called a Pythagorean tree?

   (e) Repeat this process as many times as you'd like. The true Pythagorean tree has the process repeated an infinite number of times!
2. Try it out again with another triangle.
3. Try it again using a different triangle for each stage. Unless there's a pattern to how you choose the new triangles, this is what's called a random fractal.



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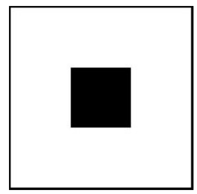
The Sierpinski Carpet

1. Making the Sierpinski carpet.
   (a) Start by drawing a large square (or find a page with one pre-drawn).
   (b) Divide the square into nine equal squares.
   (c) Shade in the center square.



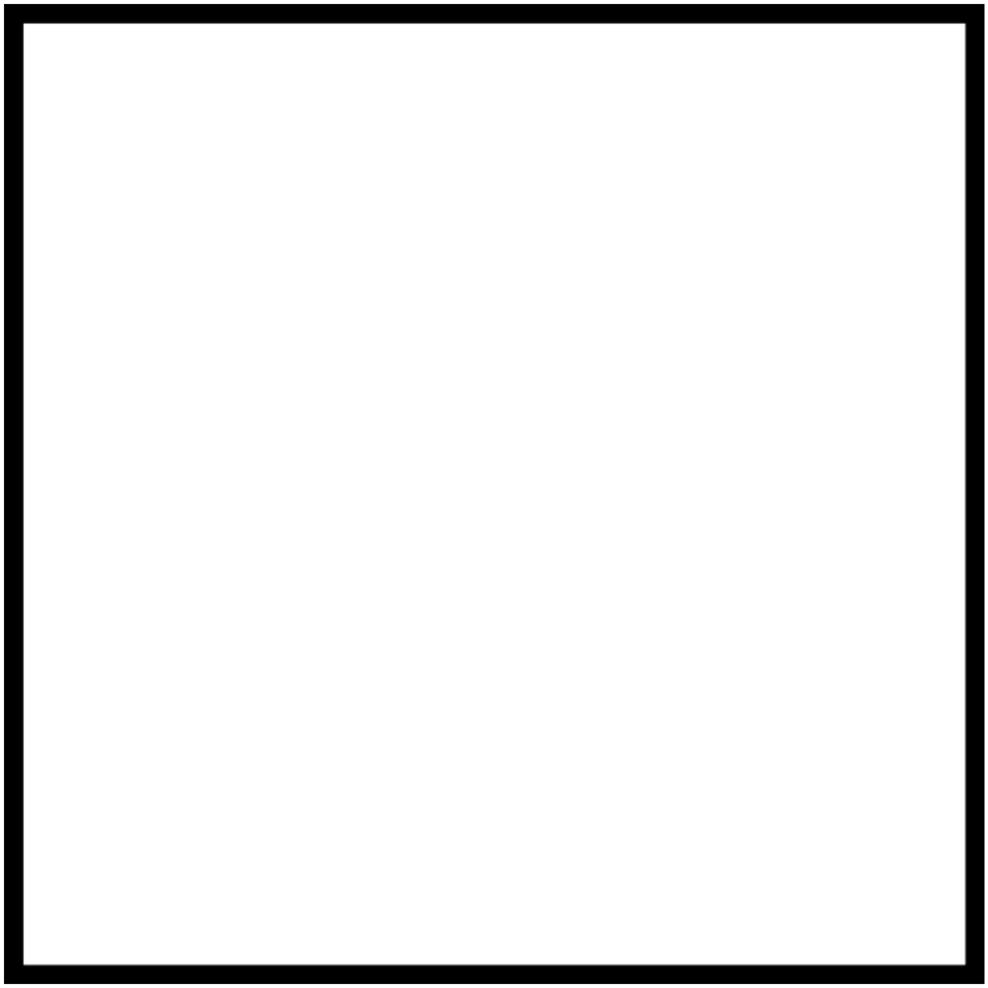
   (d) Repeat this process with each of the unshaded squares. How many unshaded squares does this give you?

   (e) Repeat this process as many times as you'd like. The true Sierpinski carpet has the process repeated an infinite number of times! You can also build a Sierpinski carpet out smaller Sierpinski carpets–how many will you need to make a carpet with three times the side length?

2. Find the total area of the unshaded squares.

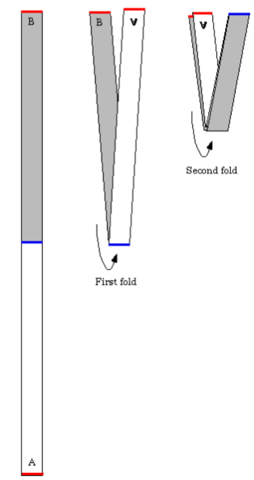
Figure	Area
Original Square	
1 Iteration	
2 Iterations	
3 Iterations	
4 Iterations	
n Iterations	

3. If you could complete this process infinitely many times, what would you expect the area of the unshaded squares to be? Would every point be shaded?

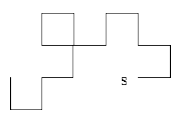


The Dragon Curve

1. Making the Dragon Curve.
   (a) Start by cutting out a long strip of paper.
   (b) Fold the strip in half.
   (c) Fold it in half three more times. Be sure to always fold in the same direction.

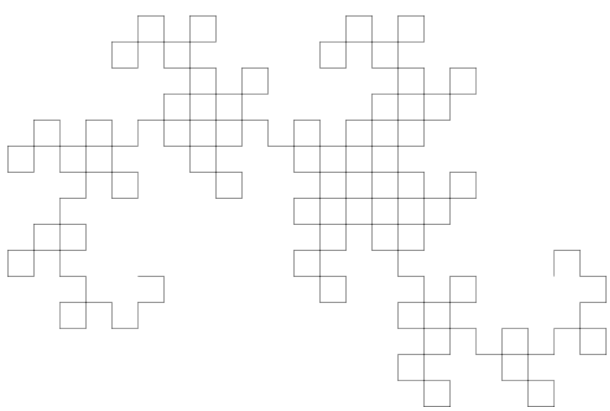


   (d) Unfold the strip, leaving each crease as a 90 degree angle. It should look something like this:



   (e) You can build a larger dragon curve out of several smaller dragon curves. Try taping together two dragons of the same size. (If you did it right, you should be able to fold the new dragon at the tape to nest the two smaller dragons into each other.) Work with your friends!

2. How big of a dragon curve can you make?



Though it's not explicitly covered, a core part of this lesson is the limits of sequences and series. On the more difficult questions, we'll be helping the students through the idea of limits without explicitly doing calculus. Have fun with infinity!

The Dragon Curve

The handout here has no questions that require answers. As the students are working, ask them how many corners there will be after a certain number of folds. The formula here is $2^n - 1$, where n is the number of folds.

The Pythagoras Tree

The fractal is so named because in each triple of squares, the area of the smaller squares sums up to the area of the largest square via the Pythagorean Theorem.

If you try drawing up some trees of your own, they will eventually overlap with themselves after some number of iterations. A tree with an isosceles triangle will be symmetric, while a tree with any other fixed ratio will curl over to one side. As for the random tree, well, see the illustration on the page for an example.

The Snowflake

This is properly called a Koch Snowflake. Depending on how you try to pronounce "Koch", this can sound rather vulgar–not the sort of ammo you want to give to middle school students. Just to be safe, avoid addressing it by its proper name. (If you don't get it, just trust me on this one.)

If s is the side length of the triangle, the original perimeter is $3s$. In every iteration, each line segment is replaced with 4 segments each one-third of the length of the original. Thus the formula for the perimeter is $3s\left(\frac{4}{3}\right)^n$, which gets infinitely large.

The Sierpinski Triangle

The number of triangles after n iterations is 3^n. (The first time this is asked is after 2 iterations, for a total of 9 unshaded triangles). The sides of each triangle are one half the length of the triangles in the previous iteration, so the formula for the perimeter is $P\left(\frac{1}{2}\right)^n$, where P is the perimeter of the original triangle. (In the equilateral case, if s is the side length, then $P = 3s$.) Thus the total perimeter of the unshaded triangles is $P\left(\frac{1}{2}\right)^n 3^n = P\left(\frac{3}{2}\right)^n$, which gets infinitely large.

If students can solve all that, ask them what they think will happen to the area of the unshaded region. (It goes to 0, of course.) If they need a hint, you can ask them what they think about the area of the shaded region. If they've already done the Sierpinski Carpet, they should hopefully make the connection between the two figures.

The Sierpinski Carpet

The number of squares after n iterations is 8^n. (The first time this is asked is after 2 iterations, for a total of 64 unshaded squares). The side length of each square is one third the length of the side in the previous iteration, so the formula for the area of each square is $s^2\left(\left(\frac{1}{3}\right)^2\right)^n = s^2\left(\frac{1}{9}\right)^n$, where s is the side length of the original square. Thus the total area of the unshaded squares is $s^2\left(\frac{1}{9}\right)^n 8^n = s^2\left(\frac{8}{9}\right)^n$, which goes to 0.

Alternatively, you can compute the area of the shaded squares. Each shaded square has area equal to that of the unshaded square for that same iteration (i.e. $s^2\left(\frac{1}{9}\right)^n$). The number of shaded squares is 8^{n-1} Thus the total shaded area after n iterations is

$$s^2\left(\frac{1}{9}\right) + s^2 8\left(\frac{1}{9}\right)^2 + s^2 8^2\left(\frac{1}{9}\right)^3 + \ldots + s^2 8^{n-1}\left(\frac{1}{9}\right)^n$$

$$= s^2\left(\frac{1}{9}\right)\left(1 + \left(\frac{8}{9}\right) + \left(\frac{8}{9}\right)^2 + \ldots + \left(\frac{8}{9}\right)^{n-1}\right) = s^2\left(\frac{1}{9}\right)\left(\frac{1-\left(\frac{8}{9}\right)^n}{1-\frac{8}{9}}\right)$$

$$= s^2\left(1-\left(\frac{8}{9}\right)^n\right)$$

As n gets larger, this gets closer and closer to s^2, which is of course the entire area of the original square. Thus the unshaded regions have area 0.

For the two other questions, a Sierpinski carpet can be built out of 8 smaller carpets that have one third of the side length. Tile them into a 3x3 grid with the center square missing. Also, even though the unshaded regions will have 0 area, not every point will be shaded: every point on one of the dividing lines between the squares will always be unshaded.

If students can solve all that, ask them what they think will happen to the perimeter of the unshaded region. (It goes to infinity, of course, though you'll have to use the geometric series again.) If they've already done the Sierpinski Carpet, they should hopefully make the connection between the two figures.