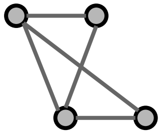
UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

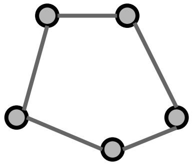
GRAPHS

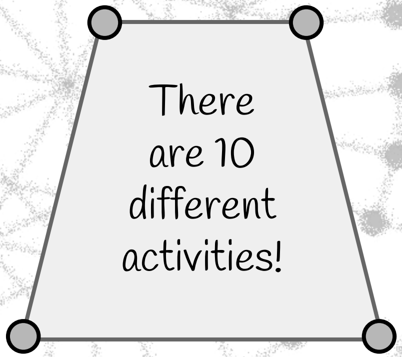
A graph G consists of a set of **nodes** (○), and some **edges** between two nodes (○—○).
If nodes x and y are directly connected by an edge, we say that x and y *are adjacent*.

Example of a Graph
- 4 nodes
- 5 edges



A "circle": Every node has two neighbors

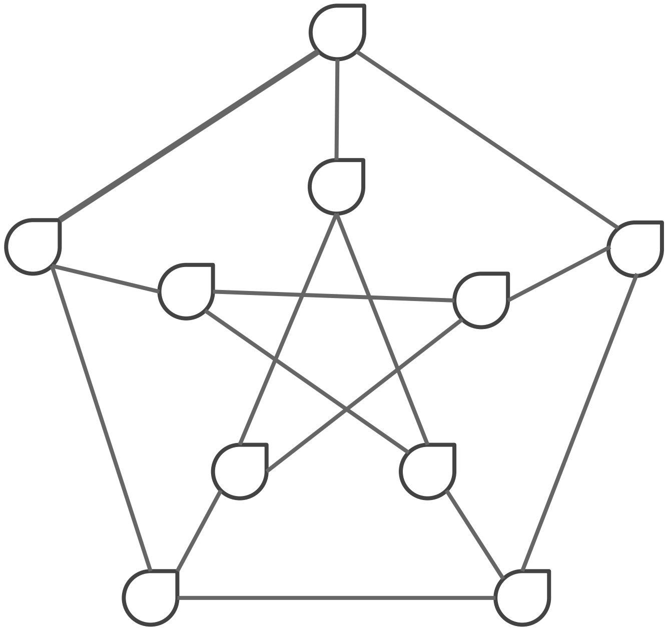




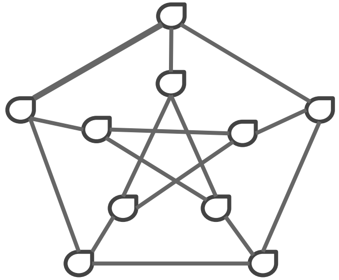
Useful information for the graphs that we will consider in the activities:

- No "loops" are allowed (so an edge must connect different nodes).
- Edges don't have any direction.
- If two nodes x and y are adjacent, we say that y is a neighbor of x (and also of course: x is a neighbor of y).
- The degree of a node is the number of neighbors that it has. It tells you "how popular" the node is.
- Note: a node **is not** its own neighbour.

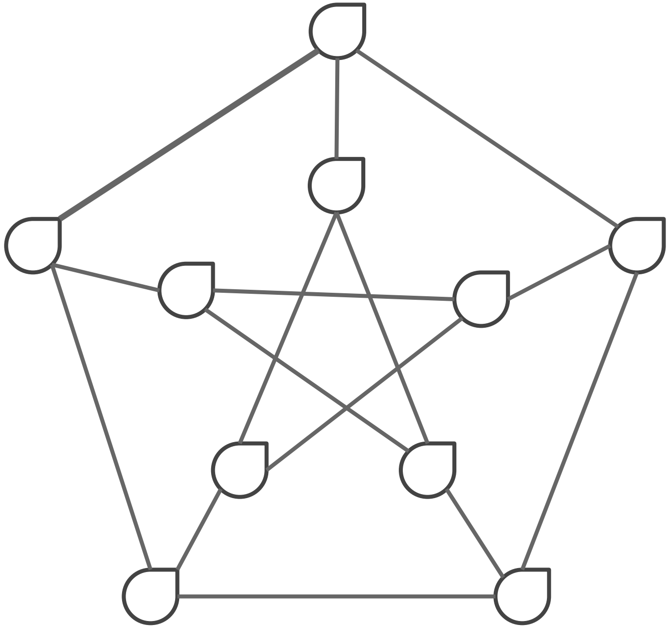
(1) Coloring nodes of the Petersen graph



? What is the smallest number of colors needed to color all the 10 nodes of the Petersen graph, so that nodes that are adjacent (meaning: directly connected by an edge) receive different colors? Color the graph.



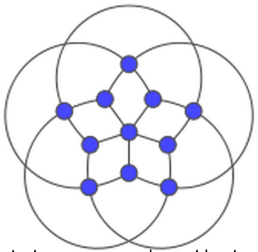
(Petersen Graph)



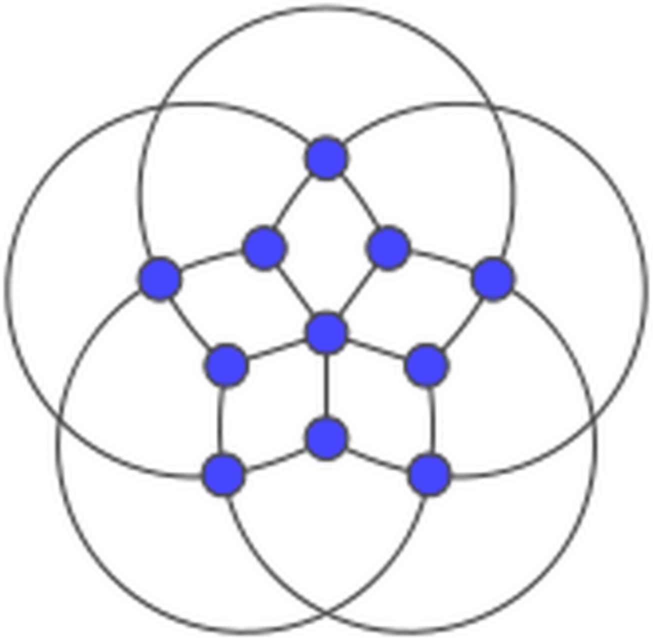
(we put various copies of the Petersen graph, if you want to experiment with different colorings)

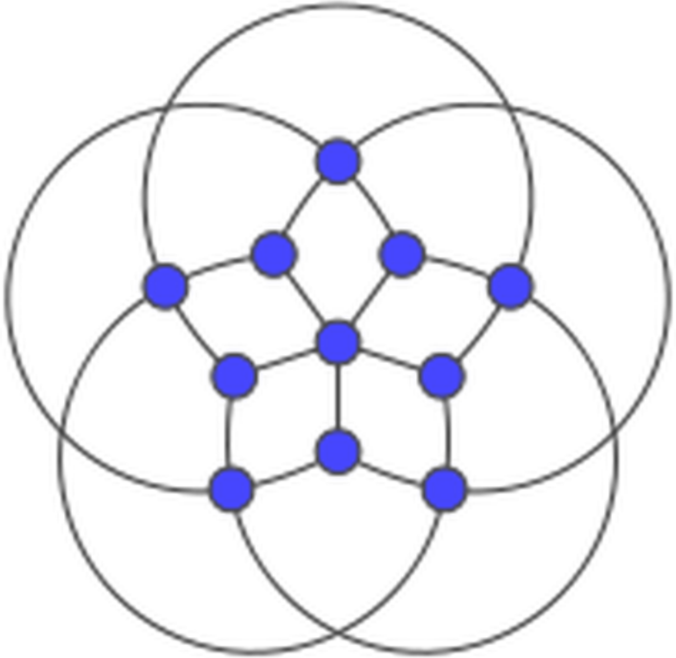
(2) Coloring the edges of Grotzsch Graph

? What is smallest number of colors needed to color the edges of the Grotzsch graph, so that edges that are adjacent have different colors? Color the graph.



[Note: remember that two edges are considered adjacent if they share one node.]

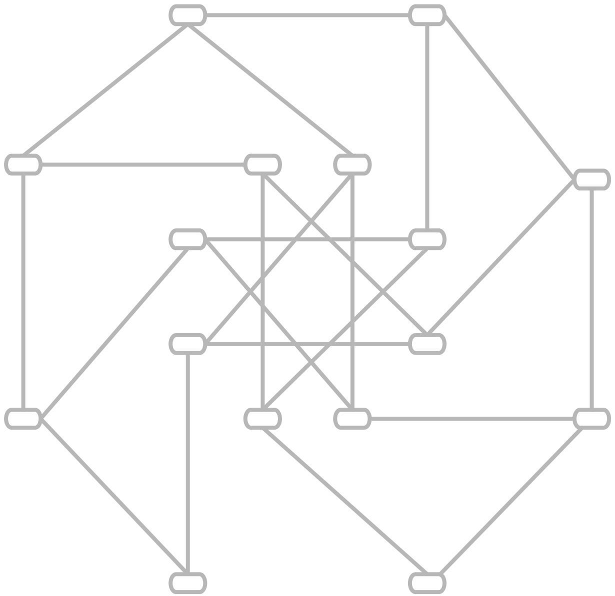


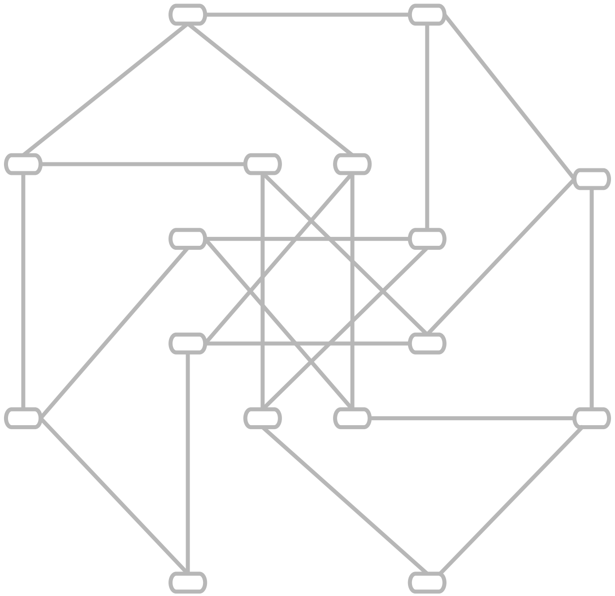


(we put various copies of the Grotzsch graph, if you want to experiment with different colorings)

(3) The Mobius-Kantor Graph

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/





(we put various copies of the Mobius-Kantor graph, if you want to experiment with different colorings)

Color all the nodes of the Mobius-Kantor graph G, so that nodes that are adjacent (directly connected by an edge) get different colors. Use the least possible number of colors.

Also, color all the edges of G, so that nodes that are adjacent (meaning: directly connected by an edge) receive different colors. Use the least possible number of colors.

[Note: there are two independent tasks to be performed. You may only do one of them if you like]

(4) A Softball tournament

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

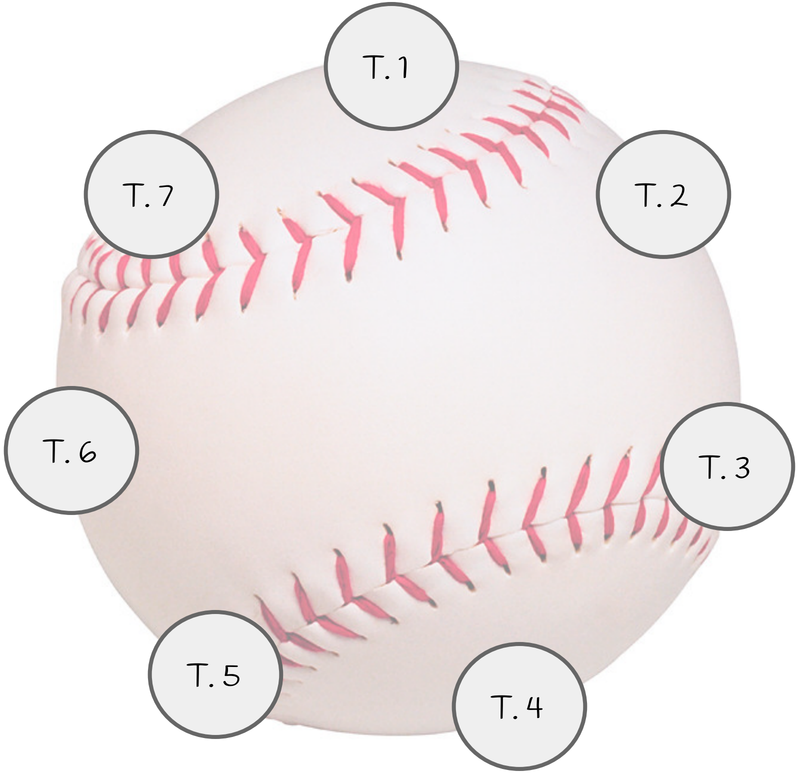
There are 7 softball teams, T1, …, T7, which will play in a tournament, so that each team plays six games (one game against every other team).

(a) How many games will be played in total?

(b) We want to find a schedule for these matches such that each team plays each of its six games on different days of the week, with no games on Sunday.

[So: a team cannot play more than one match in a single day.]

Can this be done? If so, find a working schedule, and represent your schedule as the edge-coloring of a suitable graph of 7 nodes.



Monday's games

Tuesday's games

Wednesday 's games

Thursday 's games

Friday 's games

Saturday 's games

(4) A Soccer tournament

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

There are <u>7 Soccer teams</u>, A, ..., G, which will play in a tournament, so that <u>each team plays six games</u> (one game against every other team).

(a) How many games will be played in total?

(b) Simulate all the matches using a coin (so assume no ties will occur). Draw an arrow from the match winner to the loser. (So draw one arrow per match played).

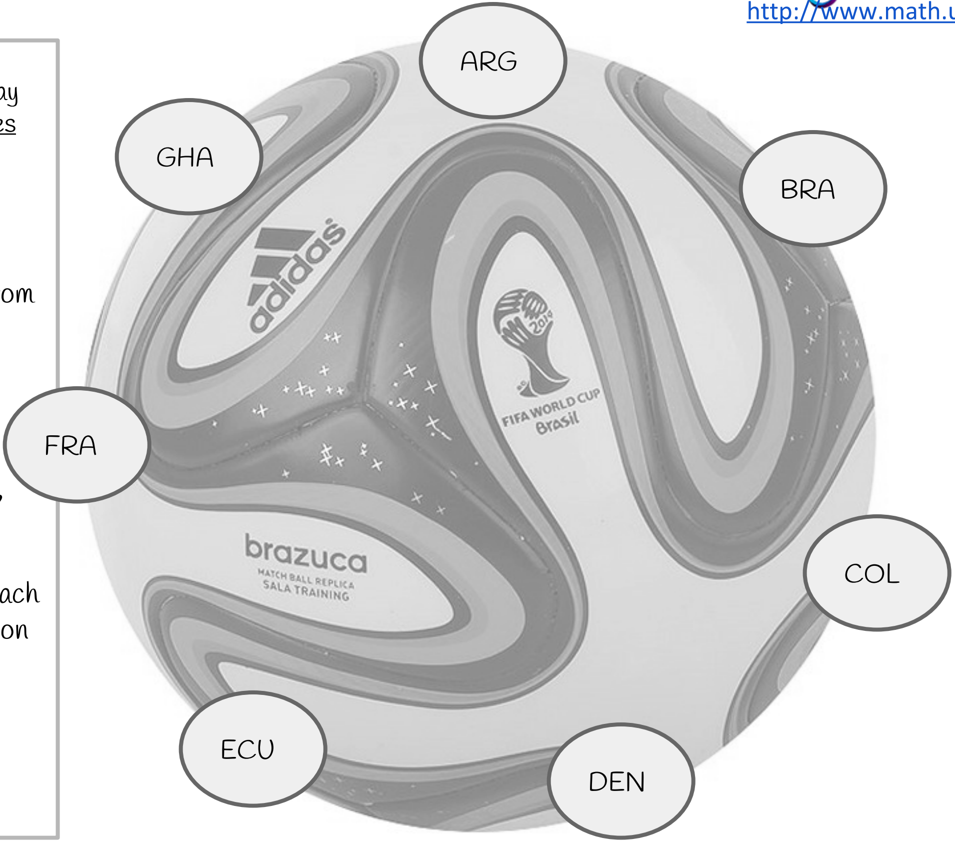
(c) Is it possible to find a "complete tour" following arrows in your graph? Discuss.

Will this happen in every single tournament? Experiment again!

[Explanation: this means traveling through each team exactly once, following the right direction of the arrows:

(1)-->(2)-->(3)-->(4)-->(5)-->(6)-->(7)

(So 1 beat 2, 2 beat 3, 3 beat 4, etc...)]



<u>Attempt of complete tour:</u>

(5) The Art Gallery

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

Consider the following "Art Gallery" with 11 nodes. We want to place several guards (possibly in the boundaries of the gallery) such that every place in the gallery is seen by at least one of them, at all times.

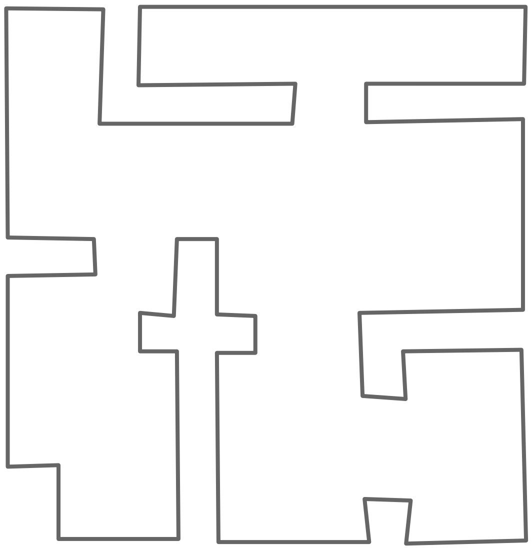
(a) Show that three guards can "cover" gallery I (read above for explanation).

[Hint: triangulate the region (make sure you do not add any nodes in the process), and then 3-color the resulting graph, such that every triangle gets all 3 colors, and adjacent nodes get different colors. Now choose the guards…]

(b) Can you use only two guards to "cover" Gallery I?

(c) What is the least number of guards required to "cover" Gallery II?

Gallery I



Gallery II

(For your information)

Chvatal's Theorem: If we have an art gallery (polygon) of n nodes, then a total of $\lfloor n/3 \rfloor$ guards (or less) can "cover" the gallery.

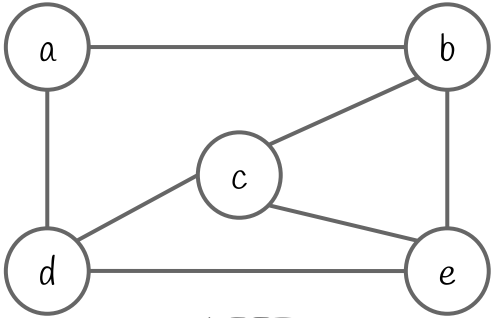
(6) THE STRUCTURE GAME

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

Given are two Graphs (LEFT and RIGHT). We describe the structure game between two players:

- Player 1 starts: he <u>selects a node</u> from one of the graphs (LEFT or RIGHT), marks it with a color, and fills the entry in the table.
- Player 2 responds selecting a node from the graph that was not used by player 1. He marks it with the <u>same color</u>, and fills the entry.
- 1st Round is complete. Round two is similar, but a <u>new color</u> must be used. Players continue this way...
- At the end of each round, check if the graph colored on the LEFT has the same edge configuration as the colored graph in the RIGHT. [Note: ignore all the non-colored nodes and also ignore edges that are next to at least one non-colored node.]
- If the configuration between the colored graphs is *not* the same, <u>Player 1 wins</u> the game in n rounds (where n is the round just played).

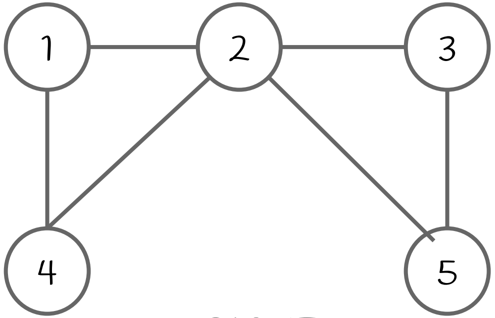
(?) Can you find a strategy for Player 1 so that he can win in the least number of rounds, no matter how player 2 plays?



LEFT

ROUNDS OF THE GAME

	1st	2nd	3rd	4th	5th
LEFT					
RIGHT					

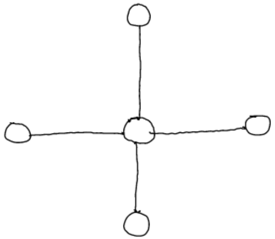


RIGHT

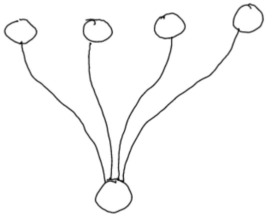
(7) Similar Graphs

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

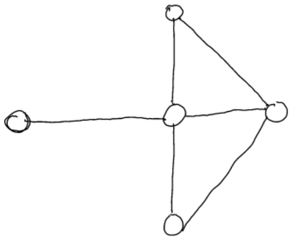
28 graphs are given. Identify all matchings of *similar* graphs. So we mean graphs that have the exact same edge configuration, after matching their nodes. (For example, graphs G1, G2 and G4 are all similar, and they are not similar to G3).
You may use colors/numbers to match the nodes. Also, you may create your own graphs!



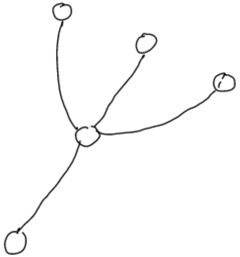
G1 Star



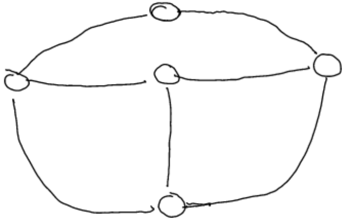
G2 Vegetation



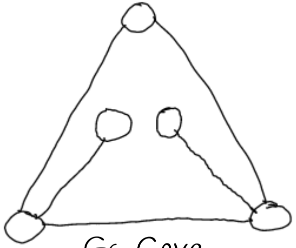
G3 Arrow



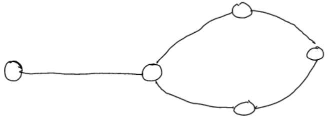
G4 Fork



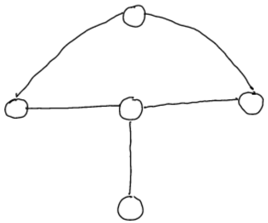
G5 Purse



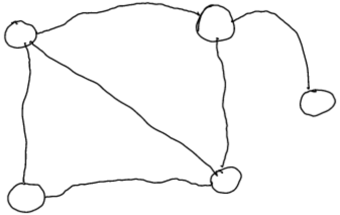
G6 Cave



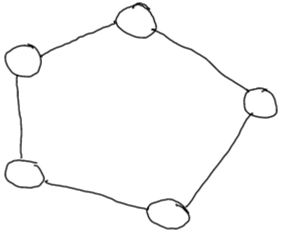
G7 Tennis Racquet



G8 Umbrella



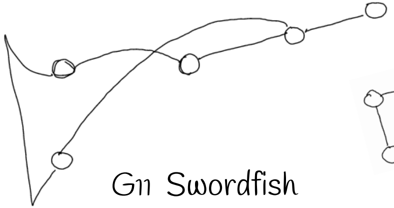
G9 Elephant

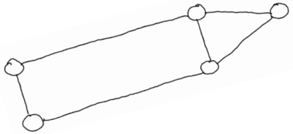


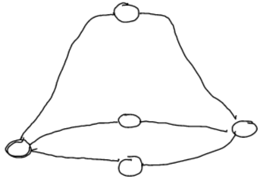
G10 Pentagon

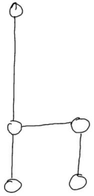
(7) Similar Graphs (...cont'd)

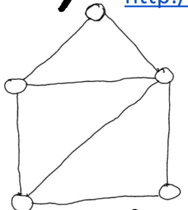
UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

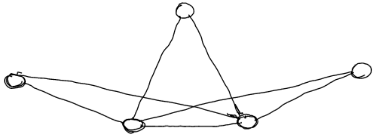

G11 Swordfish

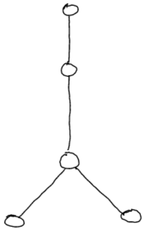

G12 Pencil

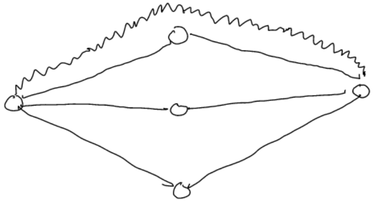

G13 Bell

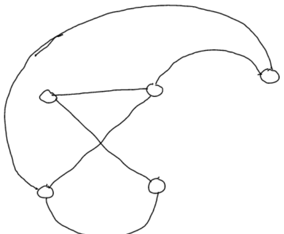

G14 Chair

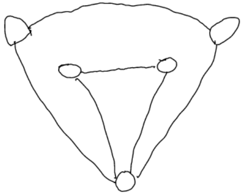

G15 House of Zorro

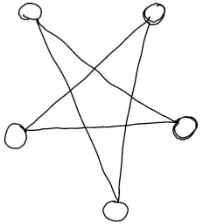

G16 Crown

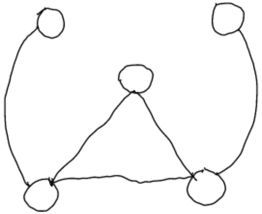

G17 Eiffel Tower

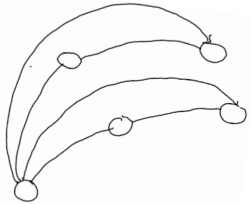

G18 Crazy Diamond


G19 Smiling Moon

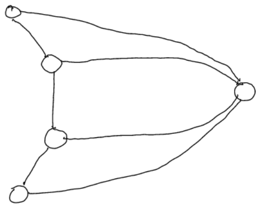

G20 Bear

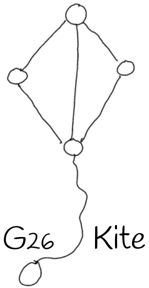

G21 Star

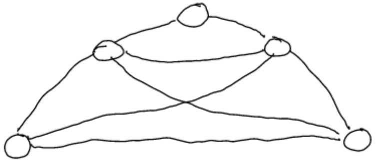

G22 Triangle Wins

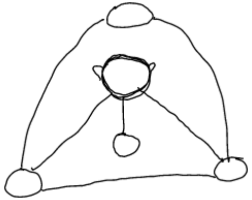

G23 Bananas

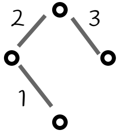
G24 __________
(make it similar to G13)


G25 Spaceship


G26 Kite


G27 Volcano


G28 Working

(8) Transforming  Edges into  Nodes

An initial graph G1 is given. Construct the graph G2 in the following way:

- The nodes of G2 are the edges of G1.
- Given two nodes in G2, connect them if they came for adjacent edges in G1 (meaning that they shared one vertex in the graph G1).

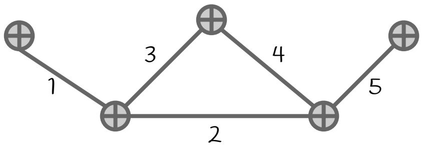
(a) How many nodes does G2 have? How many edges?

(b) Now construct G3 from G2 using the same method described above.

(c) Similarly, construct G4 from G3.

(d*) If you were to build G5 from G4, how many nodes and edges would it have?

[Note: G2 is called the *line graph* of G1]



G1

G2

G3

G4

(9) The Shrikhande Graph

UCI Math Circle 2014
http://www.math.uci.edu/~mathcircle/

Construct the following graph G:

(a) G has 16 total nodes, each node represented by a pair x=(k,n) where k=0,1,2,3 and n=0,1,2,3. Here order matters, so that for example, the nodes (1,4) and (4,1) are different nodes.

(b) Given nodes (k,n) and (r, s), connect them with and edge exactly when

$$|k-r| < 2 \text{ and } |n-s| < 2.$$

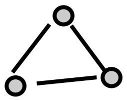
[Example: (2,3) and (1, 4) are connected. Also, (3,2) and (4,2) are connected. In contrast, (2,1) and (1,4) are not connected, since the absolute value | 1-4 | is equal to 3.]

? How many edges did you draw?

(10) Building a Graph Without Triangles

Draw a graph G such that the following three conditions are satisfied:

- G has 11 nodes.
- G is "triangle-free" (so no triangle



appears in the graph).
- If you want to color all the nodes of G, such that you give different colors to adjacent nodes, you will <u>need</u> 4 colors.