MATH CIRCLE ACTIVITY: GROUP THEORY

1. The Symmetric Groups

We call S_n the set of all possible shuffles on n cards. There is a natural way to combine shuffles, namely perform one shuffle after the other. We call this operation "composition". It is easy to check that the composition is an associative operation. Moreover, there is a trivial shuffle (the "do nothing", which does not move any card) and every shuffle has an inverse (in the sense that it can be undone). Hence the set of all shuffles with the operation of composition forms a group, which we denote by S_n.

Before we begin, we will let $\boxed{\text{n}}$ represent a card with the number n written on it.

Problem 1.1 (Counting the elements of S_n.)**. In how many ways can we shuffle n cards?**

$$\boxed{1}\,\boxed{2}\,\boxed{3}\;\cdots\;\boxed{\text{n}}.$$

Equivalently, how many ways can we rearrange the numbers $1, 2, \ldots, n$?

This will be solved using the table on the following page. Here are the instructions:

- To be accurate, fix a number to which will be the first card, and then switch around the following (see the table for an example). The first two have been done for you.
- Remember to use strategies for counting earlier arrangements to count later ones. Once you discover the rule, fill in the last row.
- For notational simplicity, we will write the notation to note the group of cards

$$\boxed{\text{m}}\,\boxed{\text{n}}\,\boxed{\text{o}} \mapsto (mno).$$

(number of cards)	possible arrangements	how many?	total number
1	Start with 1: (1)	1	1
2	Start with 1: (1 2) Start with 2: (2 1)	1 1	2 2
3	Start with 1: (1 2 3) (1 3 2) Start with 2: $(2\ \cdot\ \cdot)$ $\vdots$ $(2\ \cdot\ \cdot)$ Start with 3: $(3\ \cdot\ \cdot)$ $\vdots$ $(3\ \cdot\ \cdot)$	2 2 2	$3 \cdot 2 = 6$
4	Start with 1: $(1\ \cdot\ \cdot\ \cdot)$ Start with 2: $(2\ \cdot\ \cdot\ \cdot)$ Start with 3: $(3\ \cdot\ \cdot\ \cdot)$ Start with 4: $(4\ \cdot\ \cdot\ \cdot)$		
n			

Problem 1.2 (2-Row Notations.)**.** We denote the shuffle below by the 2-row notation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 2 & 4 \end{smallmatrix}\right)$.

$$\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}\,\boxed{6}$$
$$\downarrow$$
$$\boxed{3}\,\boxed{6}\,\boxed{1}\,\boxed{5}\,\boxed{2}\,\boxed{4},$$

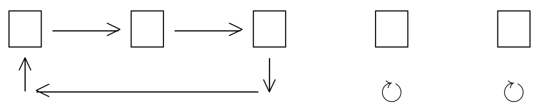
Write down all the shuffles of the 3 **cards using** 2**-row notation**.

Problem 1.3 (Shuffle orders)**.** The "easiest shuffle" we can do on the set of n cards

$$\boxed{1}\,\boxed{2}\,\boxed{3}\,\cdots\,\boxed{\text{n}},$$

is to leave them exactly as they are. We call the "do nothing operation" the *identity*.

Given a way to shuffle the cards, we say the *order* of the shuffle is the smallest number of times I need to perform the identity to go back to the identity. For example, if you have 5 cards, the operation



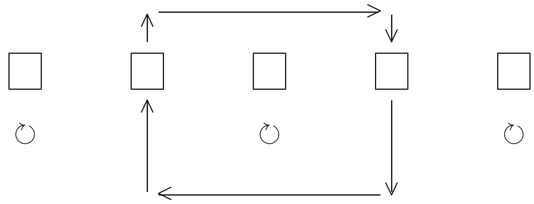
(where here ↻ means do not move this card) has order 3. To see this, look:

$$\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5} \xrightarrow[\text{do operation once}]{} \boxed{3}\,\boxed{1}\,\boxed{2}\,\boxed{4}\,\boxed{5}$$
$$\xrightarrow[\text{do operation twice}]{} \boxed{2}\,\boxed{3}\,\boxed{1}\,\boxed{4}\,\boxed{5}$$
$$\xrightarrow[\text{do operation thrice}]{} \boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}.$$

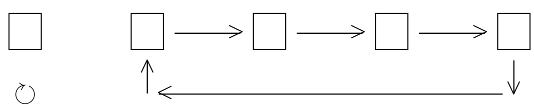
So after we do the operation 3 times, the cards are back in their original positions (as if we have operated the identity).

Compute the orders of the following shuffles.

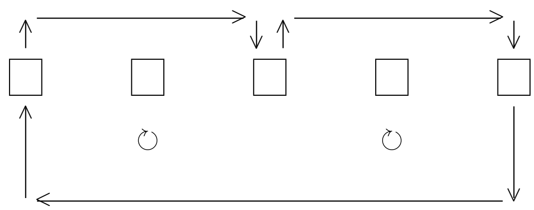
(1)



(2)

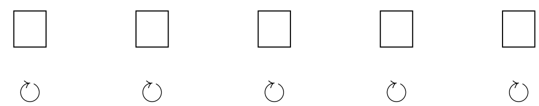


(3)



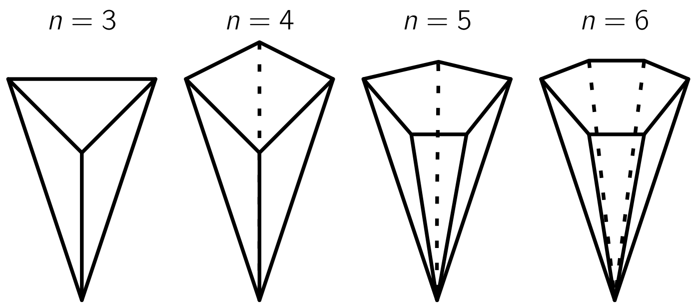
Show that if you have a total of n cards, you can find a shuffle of order 1,2,3 all the way up to n.

For instance, if you have 6 cards, draw examples of shuffles of each order. The first one has been completed for you.

Order of shuffle	Example
Order 1	
Order 2	
Order 3	
Order 4	
Order 5	
Order 6	

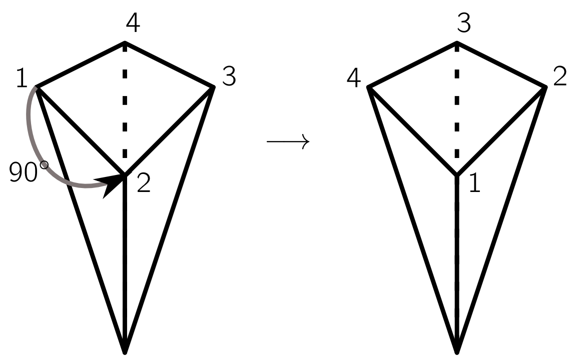
2. **The Cyclic Groups**

Problem 2.1 (The cyclic group)**.** Consider an upside down pyramid whose base is a regular polygon with n sides.



There is a trivial symmetry, which does not move the pyramid at all, and every symmetry can be undone (e.g.) to undo a 90° clockwise rotation, simply notate the pyramid 90° counter clockwise. Hence the set of symmetries of an n-pyramid forms a group, which we will call C_n, the nth cyclic group.

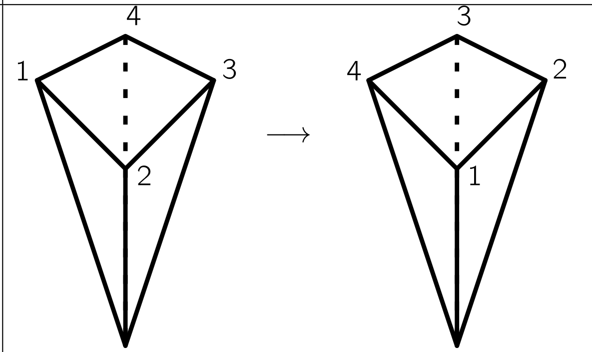
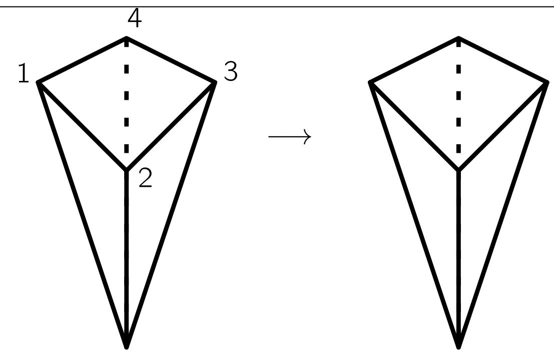
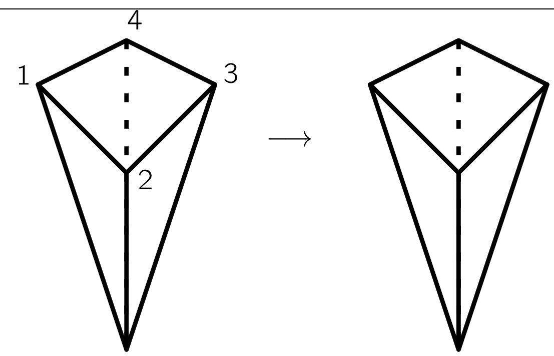
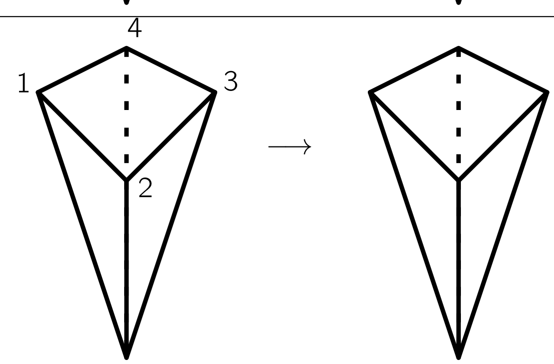
A symmetry of an n-pyramid is a rigid motion that brings the pyramid back to itself. An example is the 90° clock-wise rotation of the square pyramid:



Notice that we can combine symmetries by performing one rigid motion after another, and the result is again a symmetry of the pyramid. (result from page 5).

Note that every symmetry of the n-pyramid leads to a shuffle of the n vertices of the base of the pyramid (the polygon), hence can be described in the same 2-row notation for the shuffles of n-cards. In our previous example, the 90° clockwise rotation can be written as the shuffle $\binom{1\ 2\ 3\ 4}{4\ 1\ 2\ 3}$. If R is our basic 90° clockwise rotation, let R^6 be the symmetry resulting by rotation the pyramids 90°-clockwise 6 times. (In general, for numbers $k \geq 1$, R^k is defined similarly).

Problem 2.2 (The cyclic group C_4)**.** List the symmetries of a square pyramid. For each of the symmetries listed below, complete the picture by labeling the vertices of the base, and write down the 2-row notation. Also, write the order of the symmetry. An example is provided.

Description of the symmetry	**Diagram**	2-**row notation**	**Order**
90° clockwise rotation		$\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$	4
180° clockwise rotation		$\begin{pmatrix}1&2&3&4\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$	
270° clockwise rotation		$\begin{pmatrix}1&2&3&4\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$	
360° clockwise rotation		$\begin{pmatrix}1&2&3&4\\ \cdot&\cdot&\cdot&\cdot\end{pmatrix}$	

You may wonder whether this is the list of symmetries of the 4-pyramid.

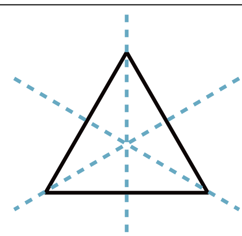
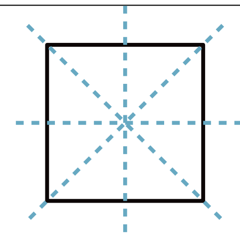
Describe R^6 **and** R^{203}**. Are they listed in the table on the previous page?** (Hopefully you will agree that our list of symmetries of the square pyramid is complete ☺).

Problem 2.3. We have noticed that every symmetry of the square can be represented in 2-row rotation as a shuffle of 4 vertices (cards).

How many possible shuffles of 4 **vertices exist? Does every shuffle rise to a symmetry of the pyramid?**

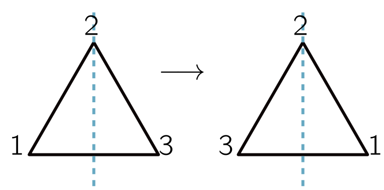
3. **The Dihedral Groups**

Problem 3.1 (The dihedral group D_n)**.** Let's now look at the symmetries of polygons. A regular n-polygon is full of symmetries. Not only can we rotate the polygon by any multiple of $\frac{360°}{n}$ (for $n = 4$, $90°$, $180°$, $270°$, $360°$), but we can also flip the polygon along a line of symmetry

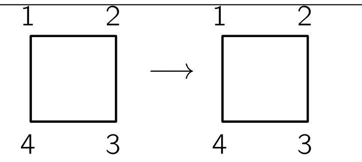
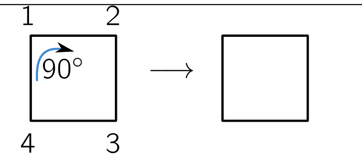
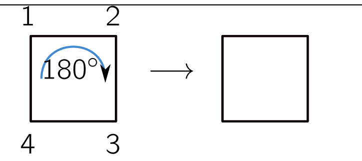
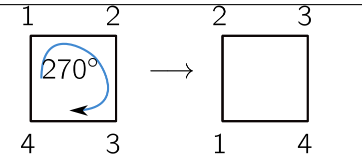
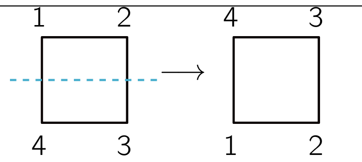
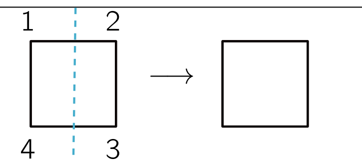
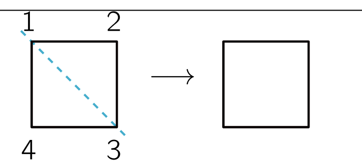
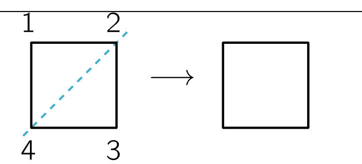
Lines of symmetry of a triangle ($n = 3$)	Lines of symmetry of a square ($n = 4$)
	

Once again, every symmetry of the n-polygon gives rise to a shuffle of the vertices hence can be described in 2-rows.

Example 3.2. The flip below can be described as the shuffle $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

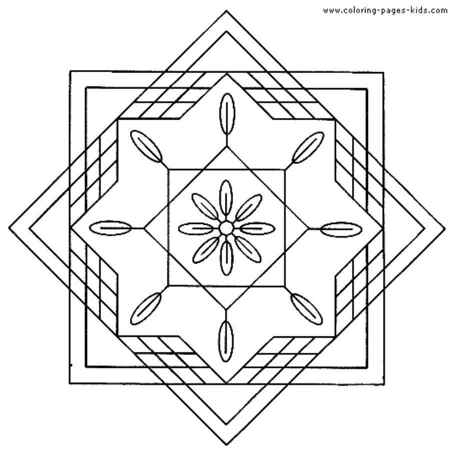


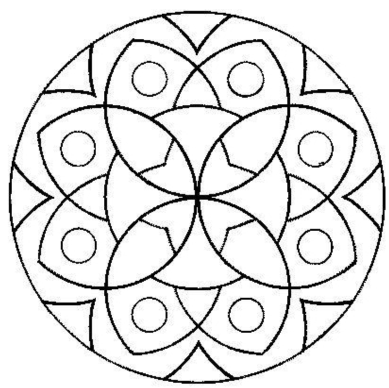
Describe the symmetries of the square. Label the vertices as appropriate, complete the 2-row notation and compute the order. A few examples are provided.

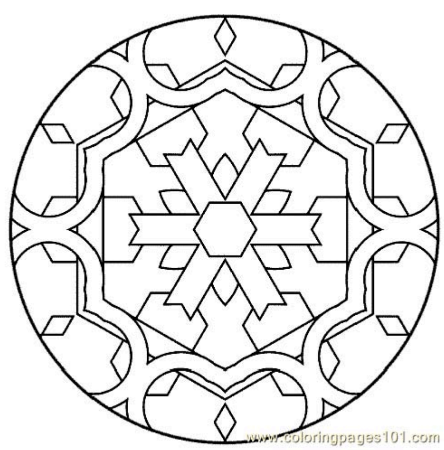
Symmetry	Picture	2-row notation	Order
"Do nothing"			1
90°-clockwise rotation		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	1
180°-clockwise rotation		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	
270°-clockwise rotation		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	
horizontal flip		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	2
vertical flip		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	2
diagonal flip		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	
other diagonal flip		$\begin{pmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \end{pmatrix}$	

Problem 3.3 (Symmetries of mandalas)**.** The symmetry groups of these mandalas are dihedral groups.

For each mandala below, find n **so that the symmetry group of the mandala is a dihedral group.**



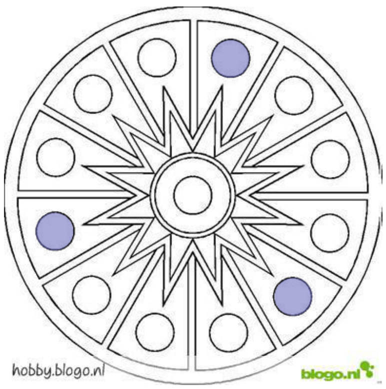


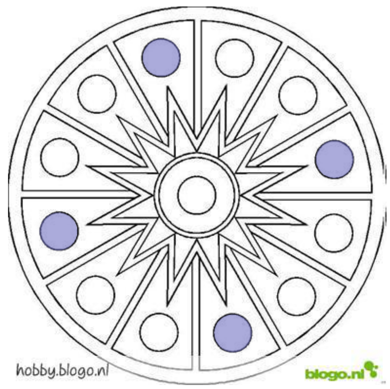


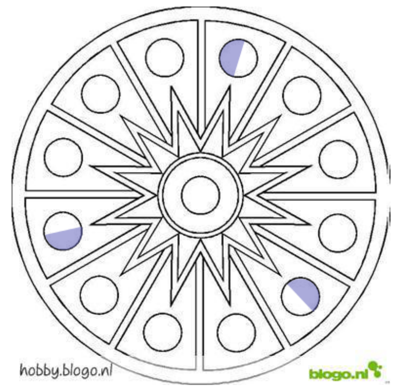
4. **Putting it all together**

Problem 4.1. The mandala below is full of symmetries (the symmetry group is of type D_{12}). We can "destroy" some of the symmetry by coloring certain parts of the mandala.

For each shaded version below, identify the symmetry group. Hint: if may be cyclic (if it only has rotations) or dihedral (if it has both rotations and flips).







Problem 4.2 (Tricky)**.** Note that the mandalas below, uncolored, have the symmetry D_{12}.

(1) Color appropriately to get C_6, C_2, D_6, D_4.

(2) Can you get C_5, C_7 or C_8? Why not?