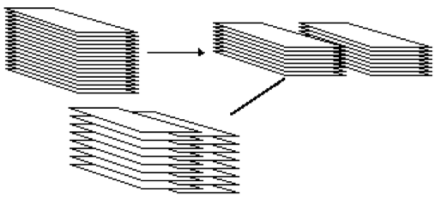
Adapted from a SFMC worksheet by D. Klein and A. Abbott

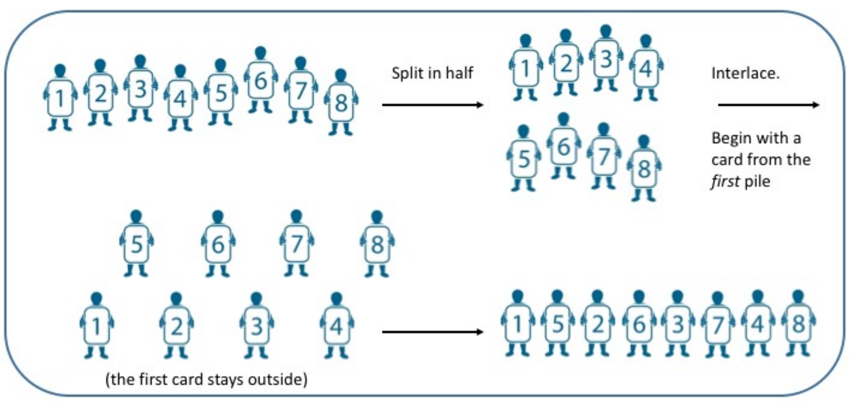
Perfect shufflings and Duplicate-Free Shuffling Chains

A perfect shuffling is a technique to shuffle a deck of cards, which consists in dividing the deck of cards into two equal piles, and perfectly interlacing them. There are 2

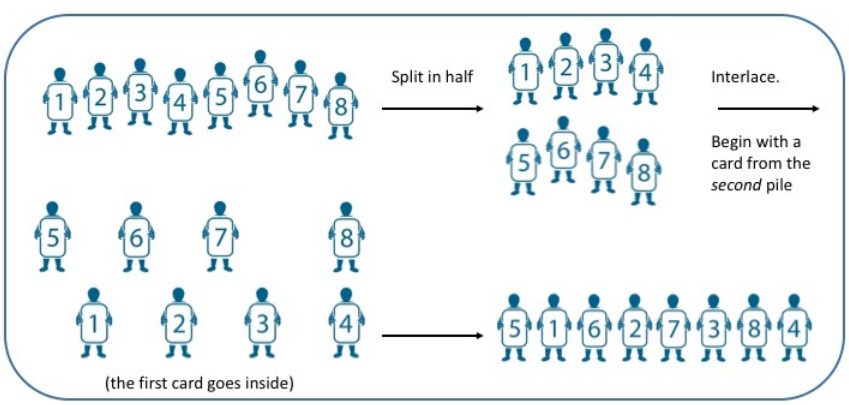


kinds of perfect shuffles (illustrated below): The **Out-shuffle** is one in which the top card remains on the outside, in the first position of the deck. The **In-shuffle** is one in which the top card moves inside, to the second position of the deck.

An OUT shuffle



An IN shuffle



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1. **Charade** Get in groups of 2, 4, 6, 8, 10, 12, . . . students, and act out the IN and OUT shuffles on a deck of 2, 4, 6, 8, 10, 12, . . . cards.

2. **Instructions** Complete the following sets of instructions:

   - To perform an **Out-shuffle**, you cut the deck exactly in half (separating them into 2 piles), then you take the . . .

   - To perform an **In-shuffle**, you cut the deck exactly in half (separating them into 2 piles), then you take . . .

3. **Diagrams** Practice drawing diagrams of IN and OUT shuffles of a deck with an even number of cards. If you want, you can perform the shuffle on the deck of cards that is provided to you before you draw the diagrams.
   Also draw the shuffle in permutation notation, as illustrated in the example.

   - **4 cards**

     $\boxed{1}\boxed{2}\boxed{3}\boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\boxed{3}\boxed{2}\boxed{4}$ OUT shuffle: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

     $\boxed{1}\boxed{2}\boxed{3}\boxed{4} \xrightarrow{\text{IN}} \boxed{}\boxed{}\boxed{}\boxed{}$ IN shuffle: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$

   - **6 cards**

     $\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6} \xrightarrow{\text{OUT}} \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$

     $\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6} \xrightarrow{\text{IN}} \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$

     OUT shuffle: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & & & & & \end{pmatrix}$; IN shuffle: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & & & & & \end{pmatrix}$

4. **Consecutive OUT shuffles . . .** We are now going to explore what is the least number of times we have to perform a perfect OUT shuffle to bring a deck of $2n$ cards back into its original configuration.
Let's start with 4 cards.

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{3}\ \boxed{2}\ \boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}$$

This was easy! To bring back a deck of 4 cards to its original configuration, I only need to perform 2 consecutive OUT shuffles.

We say that **the OUT shuffle of 4 cards has order 2**.

Does the same result holds for an OUT shuffle of 6 cards?

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{4}\ \boxed{2}\ \boxed{5}\ \boxed{3}\ \boxed{6}$$

$$\xrightarrow{\text{OUT}} \boxed{1}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{6}$$

Not quite! This is not the original ordering! If we want to restore the original configuration of a deck of 6 cards, we must apply OUT two more times:

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{4}\ \boxed{2}\ \boxed{5}\ \boxed{3}\ \boxed{6}$$

$$\xrightarrow{\text{OUT}} \boxed{1}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{6} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{3}\ \boxed{5}\ \boxed{2}\ \boxed{4}\ \boxed{6}$$

$$\xrightarrow{\text{OUT}} \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}$$

The OUT shuffle of 6 cards has order 4.

We could have reached the same conclusions using arrow diagrams, that is, keeping track of the position of each card after a shuffle is performed. The figure on the next page shows the least number of iterations of an OUT shuffle that are needed to bring each card in its original position.

YOUR TASK Find the order of the IN and OUT shuffles on 4, 6, 8, 10 and 12 cards by explicitly performing consecutive shuffles on the provided deck of cards, or by drawing shuffle diagrams on scratch paper. Complete the table on the next page.

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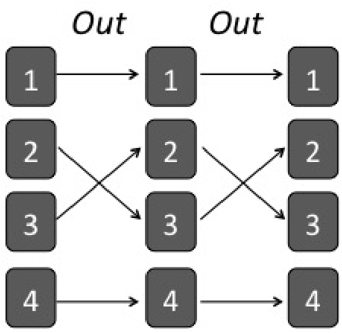
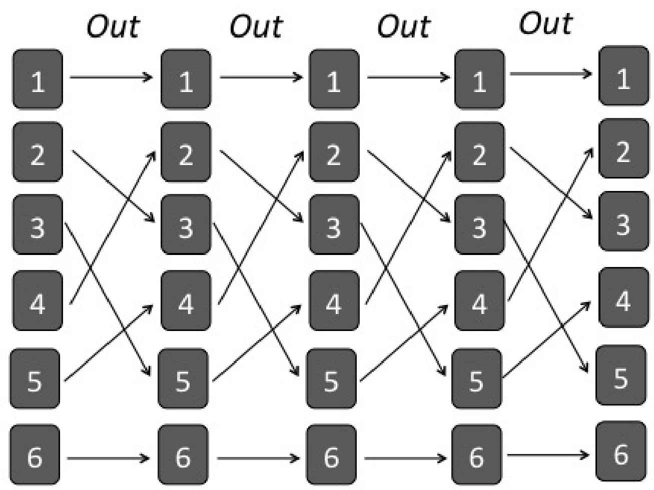




Number of cards	Order of OUT shuffle	Order of IN shuffle
$n = 4$		
$n = 6$		
$n = 8$		
$n = 10$		
$n = 12$		

Do you see any pattern?

Can you predict the order of an OUT shuffle on 14 cards?

5. **Duplicate-Free Shuffling Chains** From now on, we will work with a deck containing only 4 cards. Recall the two shuffling operations

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{3}\ \boxed{2}\ \boxed{4}$$

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4} \xrightarrow{\text{IN}} \boxed{3}\ \boxed{1}\ \boxed{4}\ \boxed{2}$$

By composing these operations (that is, applying a sequence of IN and OUT shuffles in some order) we get **shuffling chains**. An example is given below:

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{3}\ \boxed{2}\ \boxed{4} \xrightarrow{\text{IN}} \boxed{2}\ \boxed{1}\ \boxed{4}\ \boxed{3}$$

$$\xrightarrow{\text{IN}} \boxed{4}\ \boxed{2}\ \boxed{3}\ \boxed{1} \xrightarrow{\text{OUT}} \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1} \xrightarrow{\text{OUT}} \boxed{4}\ \boxed{2}\ \boxed{3}\ \boxed{1}$$

We refer to this as an OUT-IN-IN-OUT-OUT chain.

You may notice that the ordering $\boxed{4}\ \boxed{2}\ \boxed{3}\ \boxed{1}$ of the deck appears more than once in this chain. Therefore, we call it a "duplicate order" in our shuffling chain.

We are interested in **Duplicate-Free Shuffling Chains**, that is, chains with no repetitions.

Problem Is it possible to extend the following OUT-IN-OUT chain to a duplicate-Free Shuffling Chain? Try your best.

$$\boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4} \xrightarrow{\text{OUT}} \boxed{1}\ \boxed{3}\ \boxed{2}\ \boxed{4} \xrightarrow{\text{IN}} \boxed{2}\ \boxed{1}\ \boxed{4}\ \boxed{3}$$

$$\xrightarrow{\text{OUT}} \boxed{2}\ \boxed{4}\ \boxed{1}\ \boxed{3}$$

6. **The longest Duplicate-Free Shuffling Chains**

   Create a duplicate-Free Shuffling Chains which is as long as possible.

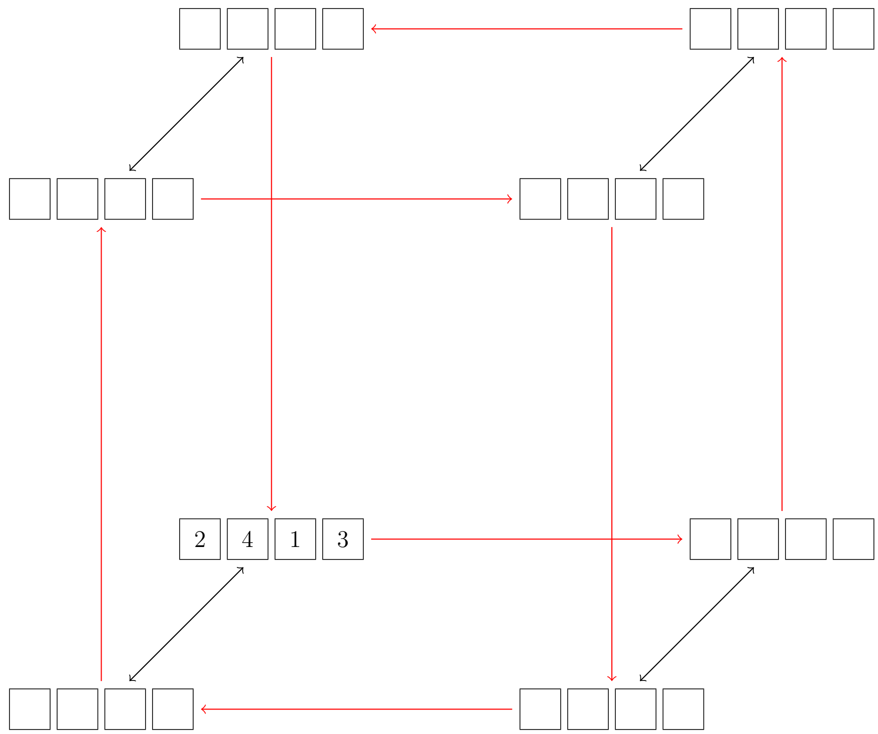
7. **How many different orderings of the 4-card deck can be created using perfect shuffles?**

   There are $4! = 24$ possible orderings of a deck with 4 cards. It makes sense to ask if we can create all of them using IN or OUT perfect shuffles, starting from the 1-2-3-4 ordering.

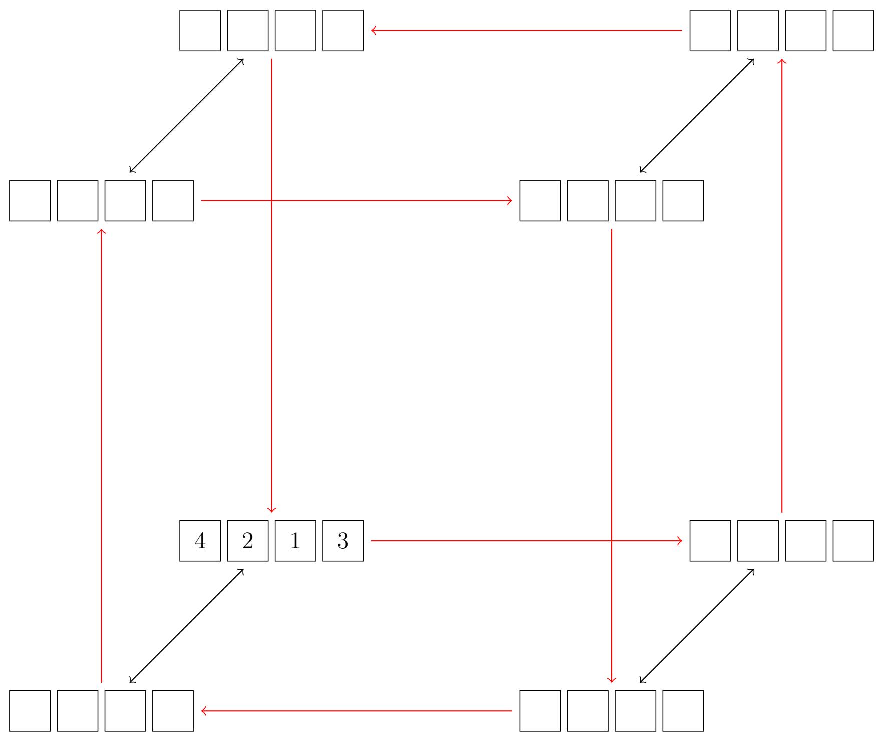
   List all the orderings that you were able to create...

8. **The Caley graph**

Complete the following graph. The red (single) arrows represent an IN shuffle. The black (double) arrows are OUT shuffles.



What if you started from 4-2-1-3? Draw another Cayley graph. How many new orderings do you get?



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This time start from 1-4-3-2. How many new orderings do you get?