Worksheet for UCI Math Cricle 01/30/2017
"Counting the Infinity"

Notations:

• Let S be a finite set. The cardinality $|S|$ is the number of elements contained in S. For example, if $S = \{1, 2, 3, 4\}$, then $|S| = 4$.

• $\mathcal{P}(S)$ is the power set. It is the collection of all subsets of S. For example, if $S = \{1, 2, 3\}$, then

$$\mathcal{P}(S) = \{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{2,3\},\ \{1,3\},\ \{1,2,3\},\ \}.$$

Here $\emptyset$ is the empty set.

• $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote sets of natural numbers, integers , rational numbers and real numbers respectively.

Definition 0.1 *A function $f : A \to B$ is called "one-to-one" (or injective) if for every $x, y \in A$,*

$$x \neq y \quad \Rightarrow \quad f(x) \neq f(y).$$

Definition 0.2 *A function $f : A \to B$ is called "onto" (or surjective) if $f(A) = B$, i.e,*

$$\{f(x)|\ x \in A\} = B.$$

Definition 0.3 *A function $f : A \to B$ is called a "bijection" if it is both one-to-one and onto.*

Countable and uncountable sets. A infinite set S is called "countably infinite" if there is a bijection between S and $\mathbb{N}$. Equivalently, elements in S can be labeled as $\{x_1, x_2, x_3, ...\}$. Otherwise, S is called "uncountable".

• "Countably infinite" is basically the smallest infinite in the sense that any infinite set contains a subset which is countably infinite.

• Examples of countably infinite sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.

• Examples of uncountable sets: $\mathbb{R}$, $\mathcal{P}(\mathbb{N})$.

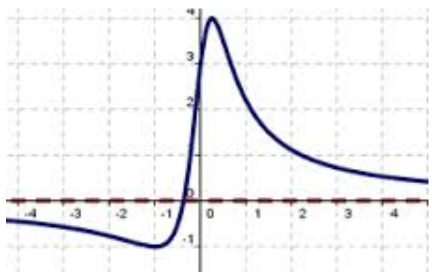
Problem 1:

(1) Draw the graph of the following function and determine whether it is a bijection

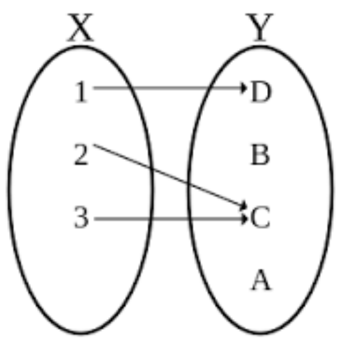
$$f(x) = \frac{x}{x-1} : \quad \mathbb{R}\backslash\{1\} \to \mathbb{R}.$$

(2) Tell whether following functions are one-to-one or onto.

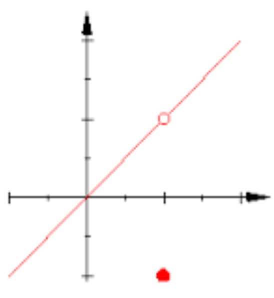
(a)



(b)



(c)



Problem 2: Suppose that S_1 and S_2 are finite sets. Assume $|S_1| = n$ and $|S_2| = m$, that is, assume that S_1 and S_2 contain n and m elements. Determine the number of functions from S_1 to S_2. If $n \leq m$, how many is one-to-one?

Problem 3: Suppose that the set S is finite and $|S| = n$. Show that

$$|\mathcal{P}(S)| = 2^n.$$

Problem 4: Find bijections between $(0, 1)$, $(0, \infty)$ and $\mathbb{R}$.

Problem 5: Denote

$$S = \{(x_1, x_2, x_3, ...) |\ \ x_i = 0 \text{ or } 1 \text{ for } i \in \mathbb{N}\}.$$

For example, $x = (1, 1, 1, 1, ...)$ and $y = (1, 0, 1, 0, 1, ...)$ are two elements of S. Find a bijection between S and $\mathcal{P}(\mathbb{N})$. Does that imply that S is uncountable? Why?

Problem 6: Suppose that S_1 is finite or countably infinite. Assume that S_2 is countably infinite. Show that $S_1 \cup S_2$ is countably infinite. *For convenience, we may assume that S_1 and S_2 are disjoint.*

Problem 7: Given that $\mathbb{R}$ is uncountable. Using Problem 6 to demonstrate that the set of irrational numbers is also uncountable.

Problem 8: Suppose that S is an infinite set and $x_0 \notin S$. Find a bijection between S and $S \cup \{x_0\}$, i.e., $|S| = |S \cup \{x_0\}|$. *This is the so called Hilbert's paradox of the Grand Hotel which illustrates that a fully occupied hotel with infinitely many rooms may still accommodate additional guests. Note that if S is countably infinite, this is an immediate consequence of Problem 6. The question is how to deal with the case when S is uncountable.*