Medium Version

1. The largest whole number such that seven times the number is less than 100 is

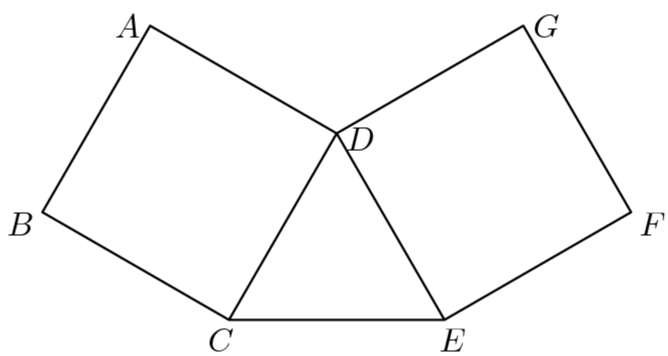
(A) 12 (B) 13 (C) 14 (D) 15 (E) 16

2. The degree of $(x^2+1)^4(x^3+1)^3$ as a polynomial in x is

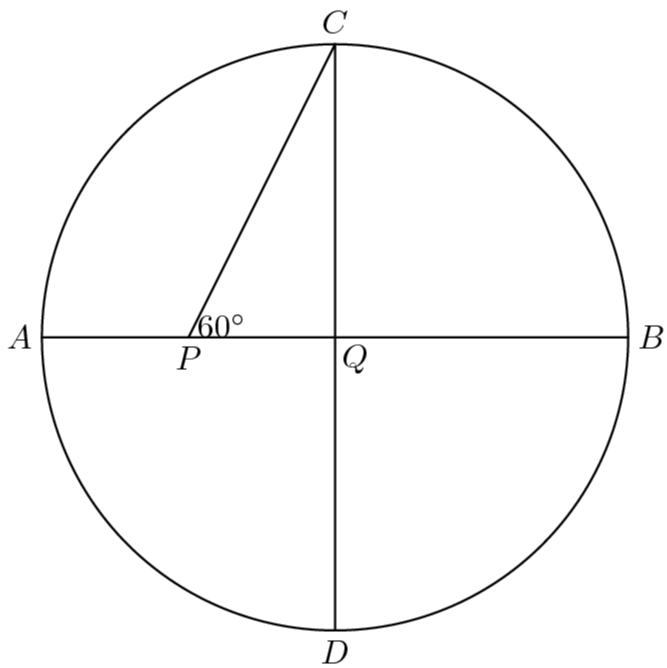
(A) 5 (B) 7 (C) 12 (D) 17 (E) 72

3. In the adjoining figure, CDE is an equilateral triangle and ABCD and DEFG are squares. The measure of $\angle GDA$ is

(A) 90° (B) 105° (C) 120° (D) 135° (E) 150°



4. If AB and CD are perpendicular diameters of circle Q, P in $\overline{AQ}$, and $\measuredangle QPC = 60^\circ$, then the length of PQ divided by the length of AQ is



(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{3}}{3}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

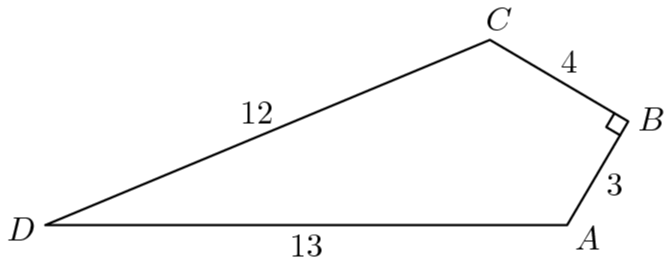
5. If the ratio of $2x-y$ to $x+y$ is $\frac{2}{3}$, what is the ratio of x to y?

(A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) 1 (D) $\frac{6}{5}$ (E) $\frac{5}{4}$

6. A positive number x satisfies the inequality $\sqrt{x} < 2x$ if and only if

(A) $x > \frac{1}{4}$ (B) $x > 2$ (C) $x > 4$ (D) $x < \frac{1}{4}$ (E) $x < 4$

7. Sides AB, BC, CD and DA of convex polygon $ABCD$ have lengths 3,4,12, and 13, respectively, and $\measuredangle CBA$ is a right angle. The area of the quadrilateral is



(A) 32 (B) 36 (C) 39 (D) 42 (E) 48

8. How many pairs (a, b) of non-zero real numbers satisfy the equation
$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$?

(A) none (B) 1 (C) 2 (D) one pair for each $b \neq 0$
(E) two pairs for each $b \neq 0$

9. A man walks x miles due west, turns $150°$ to his left and walks 3 miles in the new direction. If he finishes a point $\sqrt{3}$ from his starting point, then x is

(A) $\sqrt{3}$ (B) $2\sqrt{5}$ (C) $\frac{3}{2}$ (D) 3 (E) not uniquely determined

10. The number of teeth in three meshed gears A, B, and C are x, y, and z, respectively. (The teeth on all gears are the same size and regularly spaced.) The angular speeds, in revolutions per minutes of A, B, and C are in the proportion

(A) $x : y : z$ (B) $z : y : x$ (C) $y : z : x$ (D) $yz : xz : xy$ (E) $xz : yx : zy$

11. If the sum of the first 10 terms and the sum of the first 100 terms of a given arithmetic progression are 100 and 10, respectively, then the sum of first 110 terms is:

(A) 90 (B) -90 (C) 110 (D) -110 (E) -100

12. The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large of an angle with the horizontal (measured counterclockwise from

the positive x-axis) as does L_2, and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then mn is

(A) $\frac{\sqrt{2}}{2}$ (B) $-\frac{\sqrt{2}}{2}$ (C) 2 (D) -2 (E) not uniquely determined

13. A bug (of negligible size) starts at the origin on the coordinate plane. First, it moves one unit right to $(1, 0)$. Then it makes a 90° counterclockwise and travels $\frac{1}{2}$ a unit to $\left(1, \frac{1}{2}\right)$. If it continues in this fashion, each time making a 90° degree turn counterclockwise and traveling half as far as the previous move, to which of the following points will it come closest?

(A) $\left(\frac{2}{3}, \frac{2}{3}\right)$ (B) $\left(\frac{4}{5}, \frac{2}{5}\right)$ (C) $\left(\frac{2}{3}, \frac{4}{5}\right)$ (D) $\left(\frac{2}{3}, \frac{1}{3}\right)$ (E) $\left(\frac{2}{5}, \frac{4}{5}\right)$

14. If the function f is defined by

$$f(x) = \frac{cx}{2x+3}, \quad x \neq -\frac{3}{2},$$

satisfies $x = f(f(x))$ for all real numbers x except $-\frac{3}{2}$, then c is

(A) -3 (B) $-\frac{3}{2}$ (C) $\frac{3}{2}$ (D) 3 (E) not uniquely determined

15. A store prices an item in dollars and cents so that when 4% sales tax is added, no rounding is necessary because the result is exactly n dollars where n is a positive integer. The smallest value of n is

(A) 1 (B) 13 (C) 25 (D) 26 (E) 100

Hard Version

1. How many pairs (a, b) of non-zero real numbers satisfy the equation
$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$?

(A) none (B) 1 (C) 2 (D) one pair for each $b \neq 0$
(E) two pairs for each $b \neq 0$

2. A man walks x miles due west, turns 150° to his left and walks 3 miles in the new direction. If he finishes a point $\sqrt{3}$ from his starting point, then x is

(A) $\sqrt{3}$ (B) $2\sqrt{5}$ (C) $\frac{3}{2}$ (D) 3 (E) not uniquely determined

3. The number of teeth in three meshed gears A, B, and C are x, y, and z, respectively. (The teeth on all gears are the same size and regularly spaced.) The angular speeds, in revolutions per minutes of A, B, and C are in the proportion

(A) $x : y : z$ (B) $z : y : x$ (C) $y : z : x$ (D) $yz : xz : xy$ (E) $xz : yx : zy$

4. If the sum of the first 10 terms and the sum of the first 100 terms of a given arithmetic progression are 100 and 10, respectively, then the sum of first 110 terms is:

(A) 90 (B) -90 (C) 110 (D) -110 (E) -100

5. The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large of an angle with the horizontal (measured counterclockwise from the positive x-axis) as does L_2, and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then mn is

(A) $\frac{\sqrt{2}}{2}$ (B) $-\frac{\sqrt{2}}{2}$ (C) 2 (D) -2 (E) not uniquely determined

6. A bug (of negligible size) starts at the origin on the coordinate plane. First, it moves one unit right to $(1,0)$. Then it makes a $90°$ counterclockwise and travels $\frac{1}{2}$ a unit to $\left(1, \frac{1}{2}\right)$. If it continues in this fashion, each time making a $90°$ degree turn counterclockwise and traveling half as far as the previous move, to which of the following points will it come closest?

(A) $\left(\frac{2}{3}, \frac{2}{3}\right)$ (B) $\left(\frac{4}{5}, \frac{2}{5}\right)$ (C) $\left(\frac{2}{3}, \frac{4}{5}\right)$ (D) $\left(\frac{2}{3}, \frac{1}{3}\right)$ (E) $\left(\frac{2}{5}, \frac{4}{5}\right)$

7. If the function f is defined by

$$f(x) = \frac{cx}{2x+3}, \quad x \neq -\frac{3}{2},$$

satisfies $x = f(f(x))$ for all real numbers x except $-\frac{3}{2}$, then c is

(A) -3 (B) $-\frac{3}{2}$ (C) $\frac{3}{2}$ (D) 3 (E) not uniquely determined

8. A store prices an item in dollars and cents so that when 4% sales tax is added, no rounding is necessary because the result is exactly n dollars where n is a positive integer. The smallest value of n is

(A) 1 (B) 13 (C) 25 (D) 26 (E) 100

9. Four of the eight vertices of a cube are the vertices of a regular tetrahedron. Find the ratio of the surface area of the cube to the surface area of the tetrahedron.

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{\frac{3}{2}}$ (D) $\frac{2}{\sqrt{3}}$ (E) 2

10. Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

(A) none (B) 1 (C) 2 (D) 3 (E) 4

11. If $b > 1$, $\sin x > 0$, $\cos x > 0$, and $\log_b \sin x = a$, then $\log_b \cos x$ equals

(A) $2\log_b(1 - b^{a/2})$ (B) $\sqrt{1 - a^2}$ (C) b^{a^2} (D) $\frac{1}{2}\log_b(1 - b^{2a})$ (E) none of these

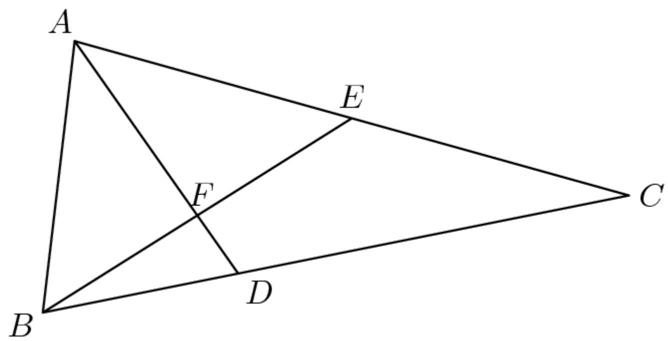
12. Let C_1, C_2 and C_3 be three parallel chords of a circle on the same side of the center. The distance between C_1 and C_2 is the same as the distance between C_2 and C_3. The lengths of the chords are 20, 16, and 8. The radius of the circle is

(A) 12 (B) $4\sqrt{7}$ (C) $\frac{5\sqrt{65}}{3}$ (D) $\frac{5\sqrt{22}}{2}$ (E) not uniquely determined

13. A box contains 2 pennies, 4 nickels, and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of coins drawn is at least 50 cents?

(A) $\frac{37}{924}$ (B) $\frac{91}{924}$ (C) $\frac{127}{924}$ (D) $\frac{132}{924}$ (E) none of these

14. In triangle ABC, $\measuredangle CBA = 72^\circ$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of triangle BDF to the area of quadrilateral $FDCE$ is



(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$ (E) none of these

15. For each real number x, let $f(x)$ be the minimum of the numbers $4x+1$, $x+2$, and $-2x+4$. Then the maximum value of $f(x)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{2}$ (E) $\frac{8}{3}$

Hardest Version

1. Four of the eight vertices of a cube are the vertices of a regular tetrahedron. Find the ratio of the surface area of the cube to the surface area of the tetrahedron.

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{\frac{3}{2}}$ (D) $\frac{2}{\sqrt{3}}$ (E) 2

2. Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

(A) none (B) 1 (C) 2 (D) 3 (E) 4

3. If $b > 1$, $\sin x > 0$, $\cos x > 0$, and $\log_b \sin x = a$, then $\log_b \cos x$ equals

(A) $2\log_b(1 - b^{a/2})$ (B) $\sqrt{1-a^2}$ (C) b^{a^2} (D) $\frac{1}{2}\log_b(1-b^{2a})$ (E) none of these

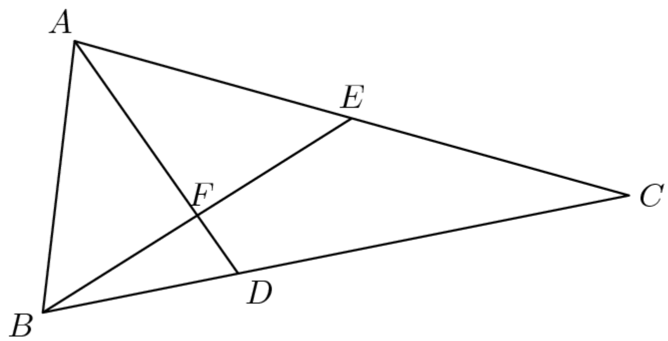
4. Let C_1, C_2 and C_3 be three parallel chords of a circle on the same side of the center. The distance between C_1 and C_2 is the same as the distance between C_2 and C_3. The lengths of the chords are 20, 16, and 8. The radius of the circle is

(A) 12 (B) $4\sqrt{7}$ (C) $\frac{5\sqrt{65}}{3}$ (D) $\frac{5\sqrt{22}}{2}$ (E) not uniquely determined

5. A box contains 2 pennies, 4 nickels, and 6 dimes. Six coins are drawn without replacement, with each coin having an equal probability of being chosen. What is the probability that the value of coins drawn is at least 50 cents?

(A) $\frac{37}{924}$ (B) $\frac{91}{924}$ (C) $\frac{127}{924}$ (D) $\frac{132}{924}$ (E) none of these

6. In triangle ABC, $\measuredangle CBA = 72^\circ$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of triangle BDF to the area of quadrilateral $FDCE$ is



(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$ (E) none of these

7. For each real number x, let $f(x)$ be the minimum of the numbers $4x+1$, $x+2$, and $-2x+4$. Then the maximum value of $f(x)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{2}$ (E) $\frac{8}{3}$

8. Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypotenuse is

(A) $\frac{4}{3}$ (B) $\frac{3}{2}$ (C) $\frac{3\sqrt{5}}{5}$ (D) $\frac{2\sqrt{5}}{3}$ (E) not uniquely determined

9. For some real number r, the polynomial $8x^3 - 4x^2 - 42x + 45$ is divisible by $(x-r)^2$. Which of the following numbers is closest to r?

(A) 1.22 (B) 1.32 (C) 1.42 (D) 1.52 (E) 1.62

10. In the non-decreasing sequence of odd integers $\{a_1, a_2, a_3, \ldots\} = \{1, 3, 3, 3, 5, 5, 5, 5, 5, \ldots\}$ each odd positive integer k appears k times. It is a fact that there are integers b, c, and d such that for all positive integers n,
$a_n = b\lfloor\sqrt{n+c}\rfloor + d,$
where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. The sum $b+c+d$ equals

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

11. Four balls of radius 1 are mutually tangent, three resting on the floor and the fourth resting on the others. A tetrahedron, each of whose edges have length s, is circumscribed around the balls. Then s equals

(A) $4\sqrt{2}$ (B) $4\sqrt{3}$ (C) $2\sqrt{6}$ (D) $1+2\sqrt{6}$ (E) $2+2\sqrt{6}$

12. The sum $\sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}}$ equals

(A) $\frac{3}{2}$ (B) $\frac{\sqrt[3]{65}}{4}$ (C) $\frac{1+\sqrt[6]{13}}{2}$ (D) $\sqrt[3]{2}$ (E) none of these

13. The polynomial $x^{2n} + 1 + (x+1)^{2n}$ is not divisible by $x^2 + x + 1$ if n equals

(A) 17 (B) 20 (C) 21 (D) 64 (E) 65

14. How many ordered triples (x, y, z) of integers satisfy the system of equations below?

$$\begin{aligned} &x^2 - 3xy + 2yz - z^2 = 31 \\ &-x^2 + 6yz + 2z^2 = 44 \\ &x^2 + xy + 8z^2 = 100 \end{aligned}$$

(A) 0 (B) 1 (C) 2 (D) a finite number greater than 2 (E) infinately many

15. A six digit number (base 10) is squarish if it satisfies the following conditions:

(i) none of its digits are zero;

(ii) it is a perfect square; and
(iii) the first two digits, the middle two digits and the last two digits of the number are all perfect squares when considered as two digit numbers.

How many squarish numbers are there?

(A) 0 (B) 2 (C) 3 (D) 8 (E) 9

Extra Problem 1:

Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

Extra Problem 2:

Consider an acute triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B+30^\circ$. Prove that $\angle A+\angle COP<90^\circ$.