Math Circles: Graph Theory

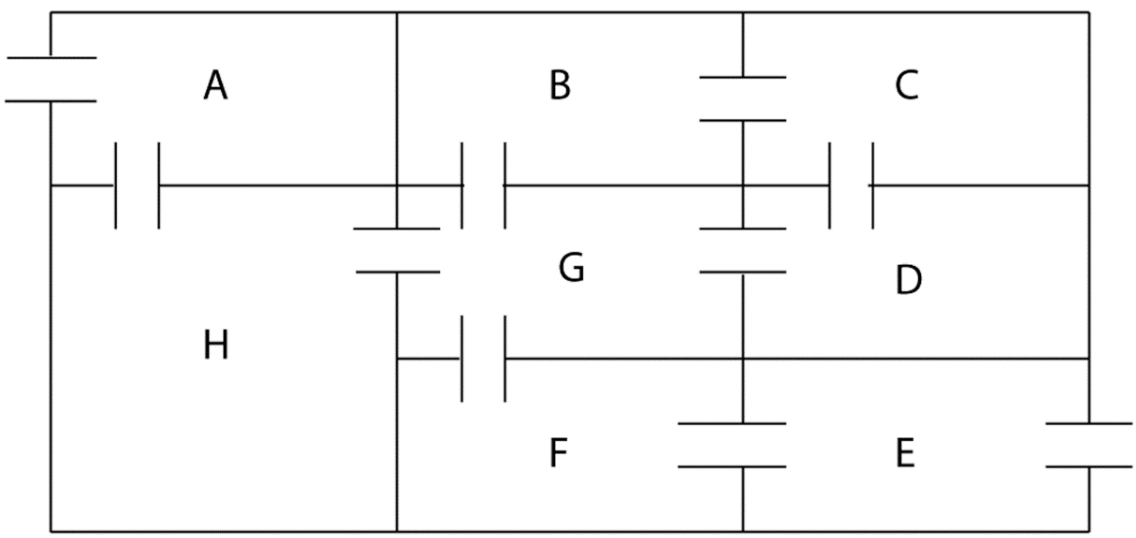
Below are several floor plans for houses. A group of friends has decided to visit each other's houses. The host's goal is to show his or her house as efficiently as possible.

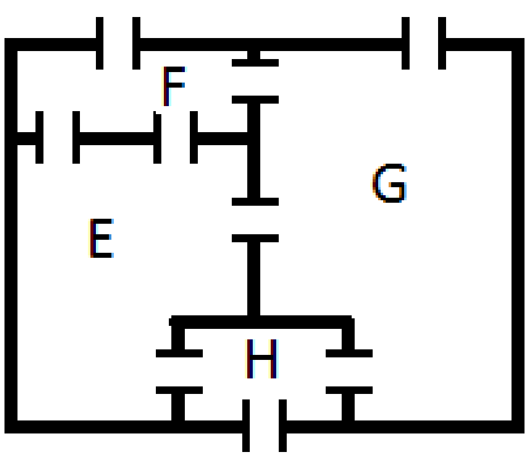
Ideally, he or she wants to start from the front door, **go through each door of the house once**, and end up at the front door again.

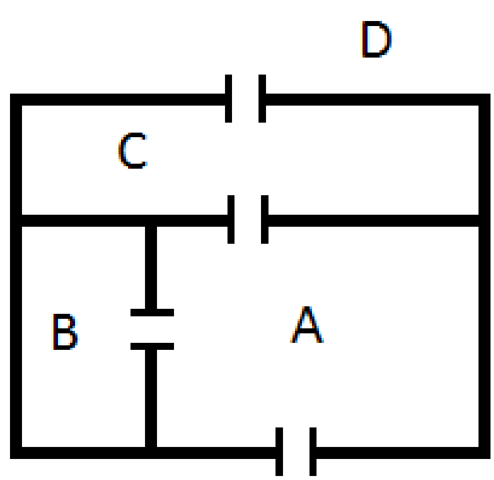
Alternatively, the host also doesn't mind if the guest start from the front door, **go through each door of the house once,** and end up in some final room.

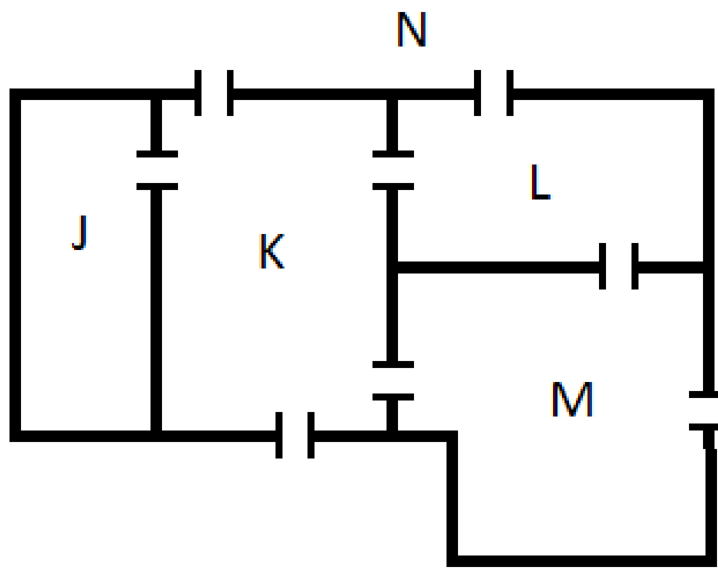
For each of the houses below, determine if either situation is possible (you can decide which door is the front door).

1. Hint: You may end up outside. That's okay, just walk back to the front.





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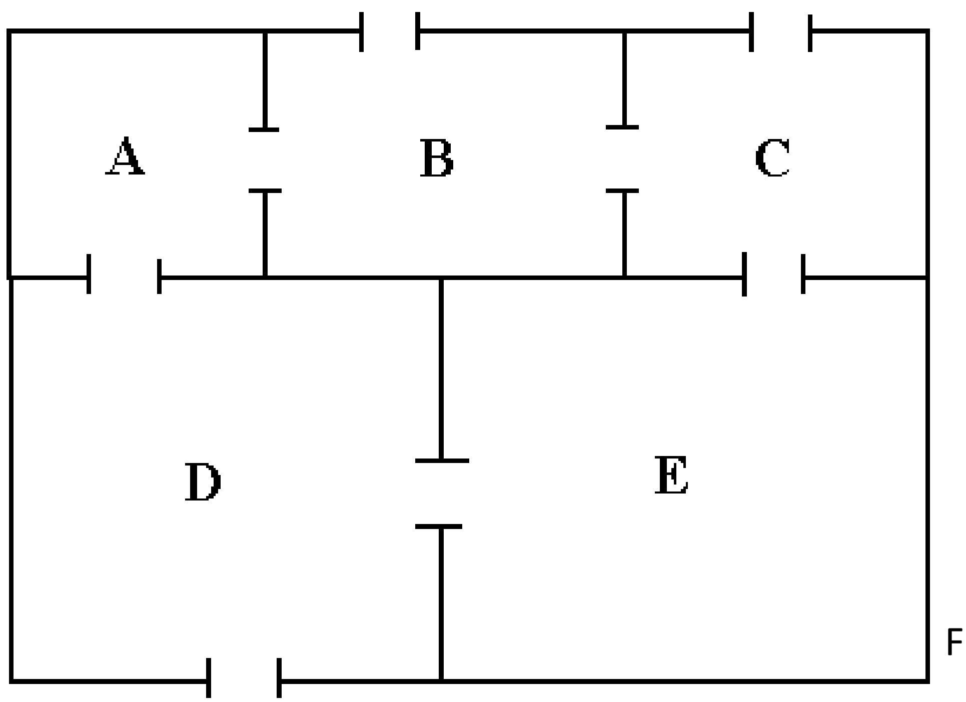
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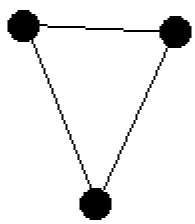
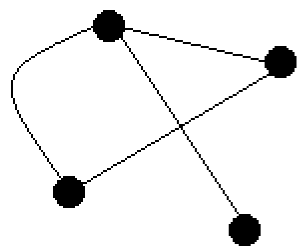
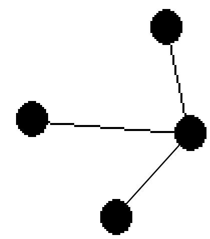
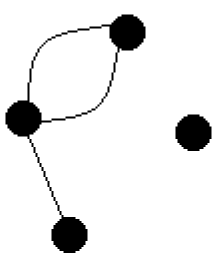
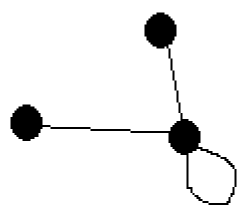
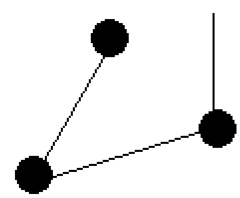
So what does this have to do with Graph Theory?

A graph is a collection of vertices and edges. You can think of a vertex as a point and an edge as a line that connects two points. The line doesn't have to be straight, but there can be only 1 line connecting two points. You also can't connect a point to itself.

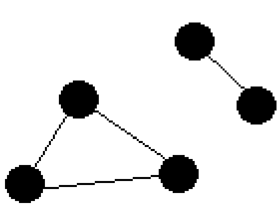
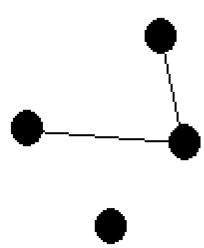
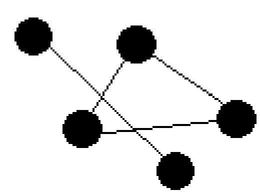
Examples of graphs:

Examples that are not graphs:

New definition: A graph is *connected* if for every pair of vertices, there is a path connecting the first vertex to the second one. In particular, there are no two completely separate graphs and there are no lonely vertices. Here are some examples of graphs that aren't connected:

Note: We will only be drawing and thinking about connected graphs today

Warm-up Exercises:

1. Draw a (connected) graph that has **4 vertices** and **3 edges**.

2. Draw a (connected) graph that has **6 vertices** and **7 edges**.

3. Draw a (connected) graph that has 5 **vertices** and **10 edges**.

Idea 1: In the last example, 10 is the maximum number of edges we can draw if we are given 5 vertices. For the next few problems, determine the maximum number of edges we can draw in a (connected) graph.

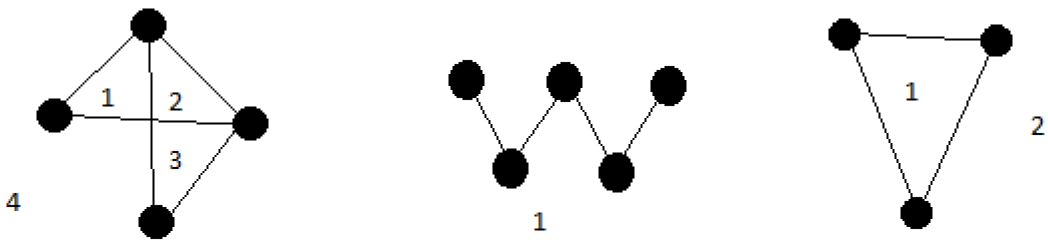
4. Maximum number of edges in a (connected) graph with **3 vertices**

5. Maximum number of edges in a (connected) graph with **4 vertices**

6. Maximum number of edges in a (connected) graph with **6 vertices**

Challenge Problem: Now that you have done a few examples, do you see a pattern? What is the formula for the maximum number of edges given *n* vertices?

New definition: A *face* is a region enclosed by lines. The outside of the graph counts as a face. Here are some examples of (connected) graphs with different numbers of faces:



Idea 2: Draw a picture of a (connected) graph with the given number of edges and vertices. Determine how many *faces* the graph has.

7. **5 edges**
   and 4 vertices

8. **3 edges**
   and 4 vertices

9. **10 edges**
   and 6 vertices

10. **7 edges**
    and 5 vertices

Challenge Problem: Do you see any pattern between the number of vertices (V), number of edges (E), and number of faces (F)? *Hint:* If you do not see the pattern right now, that is okay.

This one is trickier. Try the following exercise: Draw two more (connected) graphs for each combination of edges and vertices given above. Can you obtain a different number of faces from the one you got before, while still having a connected graph?

5 edges and 4 vertices

3 edges and 4 vertices

10 edges and 6 vertices

7 edges and 5 vertices

Euler's Formula: The number of vertices, edges, and faces are determined by Euler's Formula. Compute the following equation for each of your graphs:

$$V - E + F$$

Do you always get the same answer?

Use Euler's Formula to figure out the missing information and draw a graph for each one.

1. $F = 2$, $E = 4$

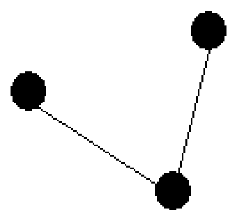
2. $V = 6$, $F = 3$

3. $E = 2$, $V = 3$

New Definition: The *degree* of a vertex is the number of edges attached to that vertex.

Idea 3: Determine the degree of each of the following vertices.

1.



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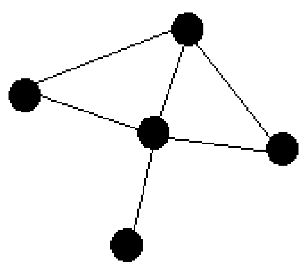


3. Draw a connected graph with 5 vertices with the following specifications:
    - 1 vertex has degree 2,
    - 4 vertices have degree 3.

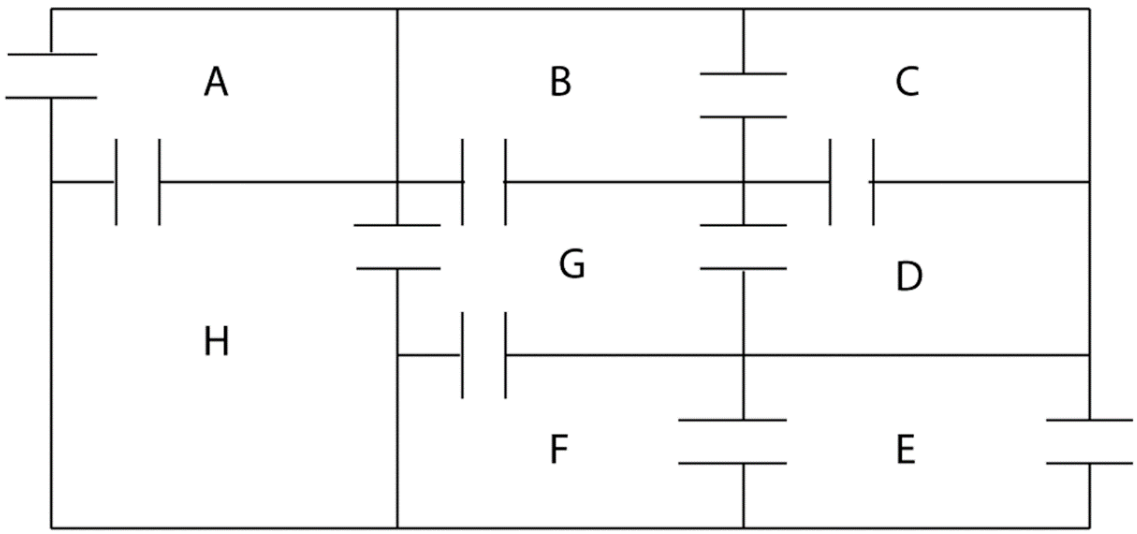
4. Draw a connected graph with 6 vertices with the following specifications:
    - 1 vertex has degree 4
    - 5 vertices have degree 2.

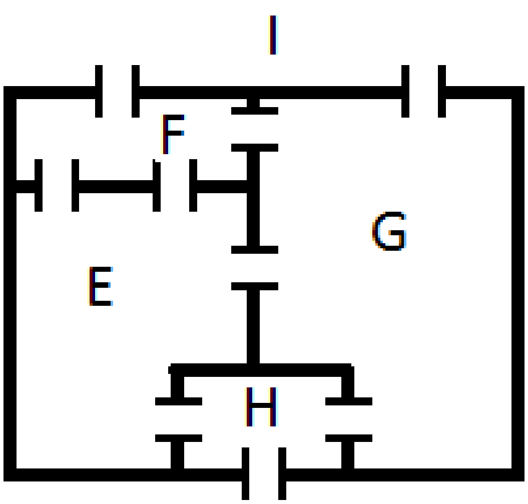
5. Draw a connected graph with 8 vertices with the following specifications:
    - 2 vertices have degree 4,
    - 6 vertices have degree 3.

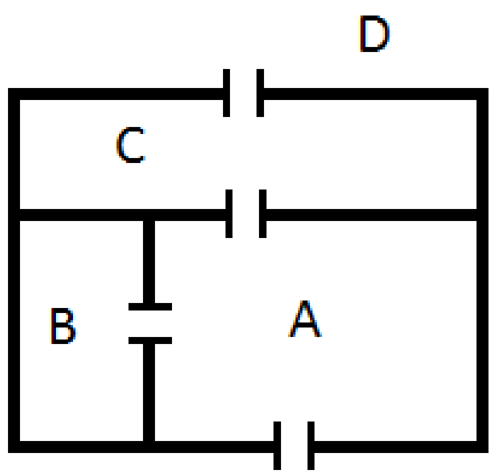
6. What is the largest degree possible for a vertex if there are *n* total vertices?

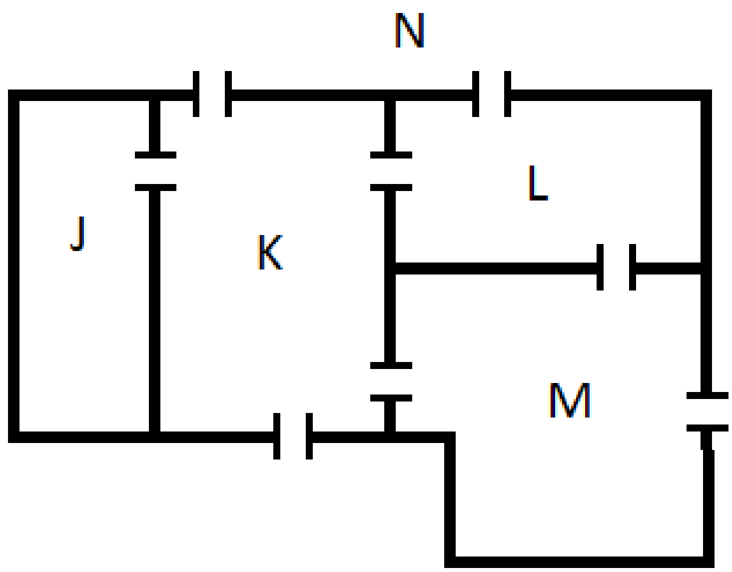
Euler Paths and Circuits

Let's go back to the original example. By drawing lines through the houses, we could figure out whether or not each host could walk through every door in the house once and end up back at the front door. How can we make this into a graph problem? For each of the floorplans on page 1, make a graph using the rooms as vertices and the doors as edges.





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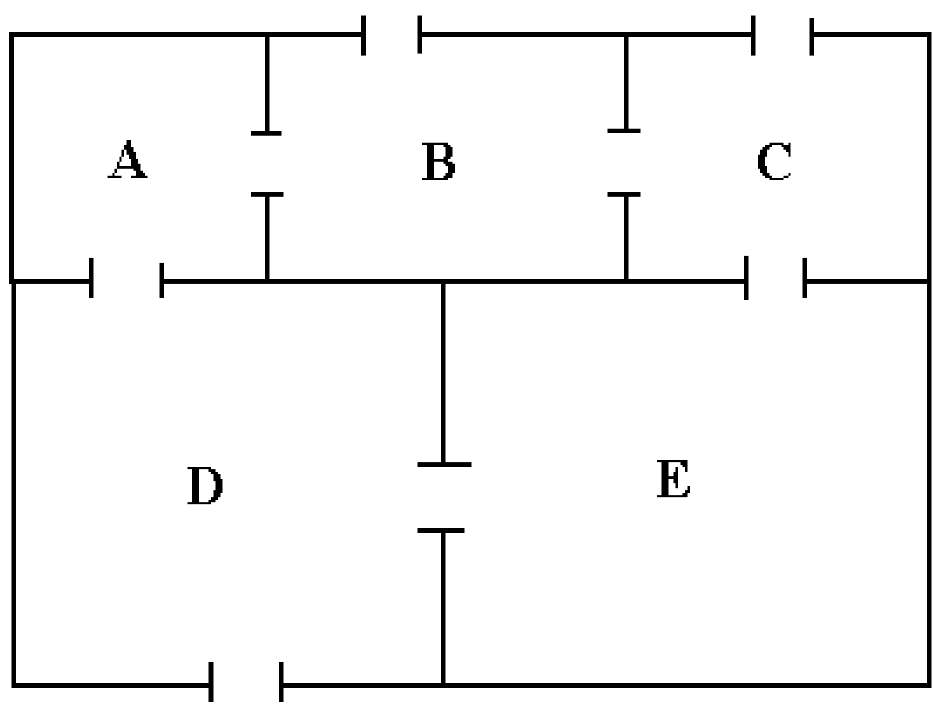
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New Definition: A *path* is a sequence of vertices connected by adjacent edges. More practically, you form a path by tracing over edges in the graph.

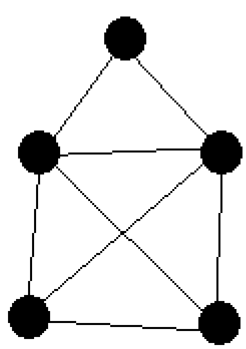
New Definition: A graph has an *Euler Path* if there is a path starting at one vertex and ending at another that uses each edge exactly once.

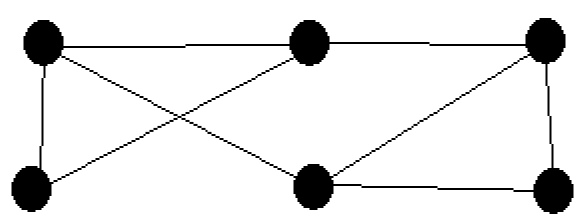
New Definition: A graph has an *Euler Circuit* if there is a path starting and ending at the same vertex that uses each edge exactly once.

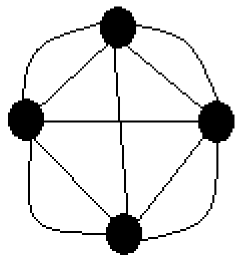
1. For each of the 5 houses, determine whether or not they have an Euler Path or Circuit.

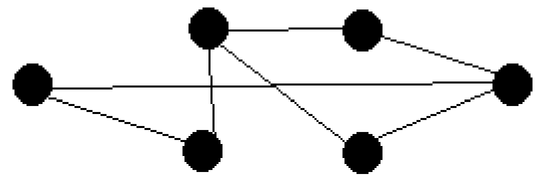
2. Count the degree of each vertex for each of the 5 houses. Is there an optimal vertex to start at based on the degree of the vertex?

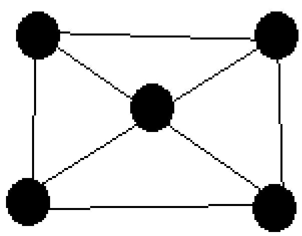
3. Determine whether these graphs have Euler Paths or Euler Circuits

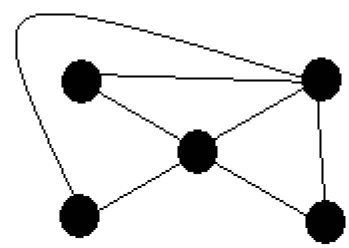












4. Do you see a pattern in the degrees of vertices when there is a Euler Path?

5. Do you see a pattern in the degrees of vertices when there is an Euler Circuit?

Challenge Problem: Can you find the pattern? When do graphs have Euler paths or circuits?