The Geometric Mean and the AM-GM Inequality

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1 Introduction:

The arithmetic mean of n numbers, better known as the average of n numbers is an example of a mathematical concept that comes up in everday conversation. We hear about averages during weather reports and teachers tell us the class average on the most recent test. However, there are other means besides the arithmetic mean. A less commonly known mean is the geometric mean. Tonight we will investigate the geometric mean, derive the arithmetic mean-geometric mean (AM-GM) inequality and do challenging problems.

2 Review of the Arithmetic Mean and Introduction to the Geometric Mean

The arithmetic mean (AM) of n numbers, better known as the average of n numbers, is the most common type of mean and it is defined by

$$AM(a_1, ..., a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The geometric mean (GM) of n numbers is the n^{th} root of the product of n numbers; that is,

$$GM(a_1, ..., a_n) = \sqrt[n]{a_1 \cdots a_n}.$$

Example 2.1 *Calculate the arithmetic and geometric mean of 2, 4, and 8.*

$$AM(2,4,8) = \frac{2+4+8}{3} = \frac{14}{3} = 4.\overline{6}.$$

$$GM(2,4,8) = \sqrt[3]{2 \cdot 4 \cdot 8} = \sqrt[3]{64} = 4.$$

Problem 2.2 *Fill in the missing blanks in the following table. In the last column write down whether the arithmetic mean or the geometric mean is greater. The first row of the table has been done as an example.*

	Arithmetic Mean (AM)	**Geometric Mean (GM)**	**AM or GM bigger?**
(8, 8)	8	8	Equal
(4, 16)			
(2, 32)			
(4, 4, 4)			
(2, 4, 8)			
(1, 9)			
(2, 8)			
(5,5)			
(2^3, 3^3, 5^3)			

We can learn a few interesting things from the previous table. For the means we calculated, the arithmetic mean was always bigger than the geometric mean except when all of the numbers being averaged were the same; in that case, the arithmetic mean and geometric mean were the same. Now, let's show that the latter observation holds in general:

Problem 2.3 *Show that if* $a_1 = \cdots = a_n = a$, *then* $GM(a_1, ..., a_n) = AM(a_1, ..., a_n)$.

It is also true in general that for all positive numbers $a_1, ..., a_n$ (not necessarily equal) that $GM(a_1, ..., a_n) \leq AM(a_1, ..., a_n)$. This inequality is called the arithmetic mean-geometric mean inequality and we will prove that it is true in the next section.

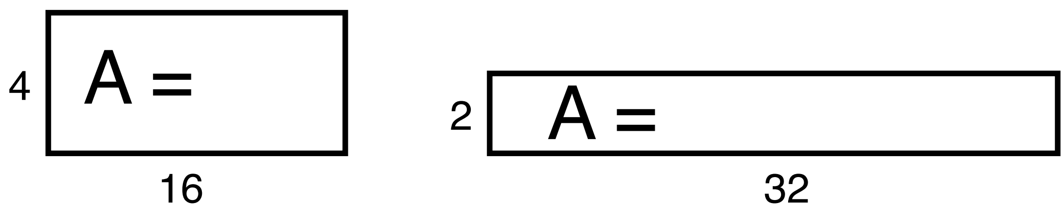
In the previous table, you found that $GM(2^3, 3^3, 5^3) = 30$. It is also true that $GM(2^2 \cdot 3, 3^2 \cdot 5, 5^2 \cdot 2) = 30$ because

$$(2^2 \cdot 3)(3^2 \cdot 5)(5^2 \cdot 2) = 2^3 \cdot 3^3 \cdot 5^3.$$

Problem 2.4 *Can you come up with another set of numbers* (a, b, c) *so that* $GM(a, b, c) = 30$*?*

At this point you may be wondering what is so geometric about the geometric mean? Let's give one possible explanation.

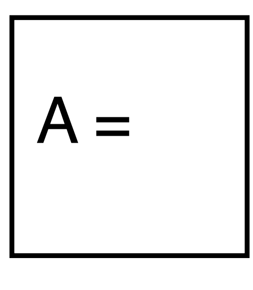
Problem 2.5 *Consider the following rectangles. Write down the area, A, of the rectangles and the geometric means of the side lengths:*



GM(4, 16) =

GM(2, 32) =

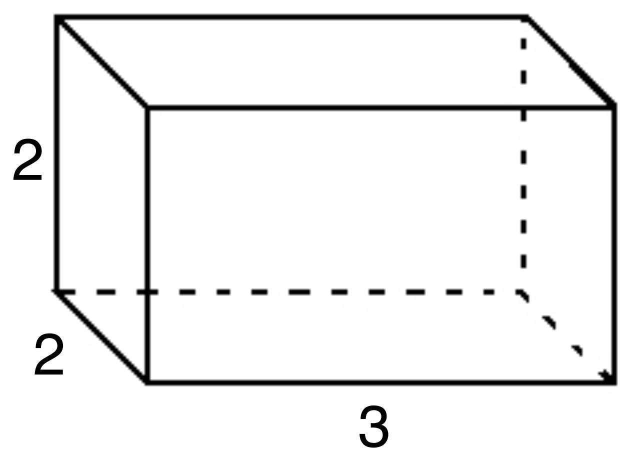
Problem 2.6 *What is the side length of the square that has the same area as the rectangles in the previous problem? Write down the side length of the square and the area of the square below.*



It looks like that given a rectangle, the geometric mean of the side lengths is the side length of the square that has the same area as the rectangle.

Is this true? Is this also true for 3-dimensional rectangles (called rectangular prisms) and 3-dimensional squares (called cubes)?

Problem 2.7 *Given the rectangular prism shown below, show that the geometric mean of the side lengths equals the side length of the cube that has the same volume as the rectangular prism.*



Normally, we only deal with 2-dimensional or 3-dimensional rectangles because humans can only see in 3-dimensions. However, we can define an n-dimensional rectangle as a rectangle with side lengths a_1, a_2, ..., a_n. The volume of the n-dimensional rectangle is $a_1 \cdot a_2 \cdots a_n$.

Problem 2.8 *Given an n-dimensional prism with side lengths $a_1, ..., a_n$ show that $GM(a_1, ..., a_n)$ equals the side length of the n-dimensional cube with the same n-dimensional area as the prism.*

We now have a geometric meaning for the geometric mean; the geometric mean of $a_1, ..., a_n$ gives the side length of the square that has the same area/volume as the rectangle with side lengths $a_1, ..., a_n$.

Now let's return to our previous question of why the geometric mean of n positive numbers is less than the arithmetic mean of the same n positive numbers.

3 AM-GM inequality

In this section we will do some fun problems that use the AM-GM inequality and prove a simplified version of the inequality. The AM-GM inequality is stated below:

AM-GM Inequality

If $a_1, ...a_n$ are positive numbers, then

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}$$

and equality holds if and only if $a_1 = \cdots = a_n$.

Use the AM-GM inequality to solve the following four problems.

Problem 3.1 *The condition that all of the a_n's be positive is important. For n odd, give an example where some of the a_n's are negative and the AM-GM inequality does not hold.*

Problem 3.2 *Use the AM-GM inequality to show that for $a, b > 0$, $\frac{a}{b} + \frac{b}{a} \geq 2$. (Hint: Apply the AM-GM inequality to $\frac{a/b + b/a}{2}$ first).*

Problem 3.3 *If* $xyz = 27$ *and* x, y, z *are all positive, then what is the minimum value of* $x + y + z$*. For what values of* x, y, z *does this minimum occur?*

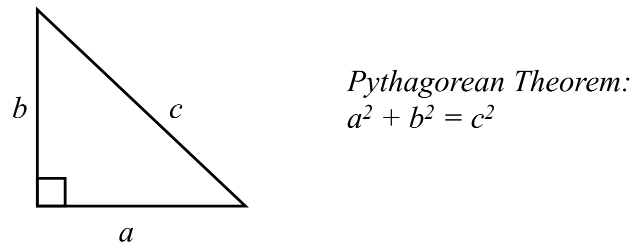
Problem 3.4 *Let* $r_1, ..., r_n$ *be* n *positive numbers and let* $x > 0$*. Show that*

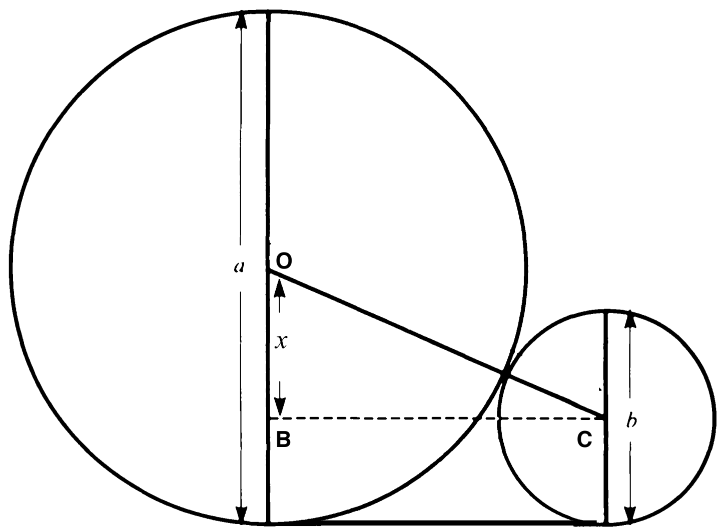
$$(x + r_1)(x + r_2) \cdots (x + r_n) \leq \left(x + \frac{r_1 + \cdots + r_n}{n}\right)^n.$$

(Hint: Take the n^{th} *root of both sides first.)*

The AM-GM inequality allows us to do cool problems like the ones you just did. Now let's investigate some proofs of the AM-GM inequality. When $n = 2$, we can give a geometric proof of the AM-GM inequality. For the next two problems you need to remember the Pythagorean theorem: If a, b, and c are the side lengths of a right triangle and c is the length of the hypotenuse of the triangle, then $a^2 + b^2 = c^2$.

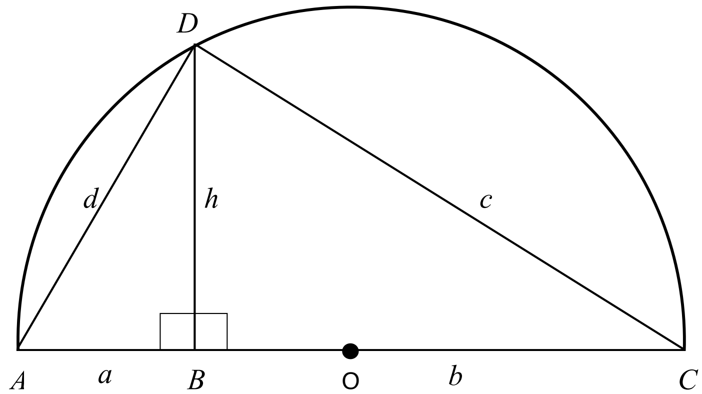
Consider the two circles below with diameters a and b with $a > b$ shown on the next page:





Problem 3.5 *Determine the length of* $\overline{OC}$*, the length of* $\overline{BC}$*, and* $\overline{OB} = x$ *in terms of* a *and* b*. Conclude that when* $a \neq b$*, that* $\sqrt{ab} < \frac{a+b}{2}$*.*

Now consider the following triangle inscribed in the semicircle with center O below:



Problem 3.6 *Let r be the radius of the semicircle. Express r in terms of a and b. Is $h \leq r$ or $r \leq h$?*

Problem 3.7 *There are three right triangles in the picture above. For each triangle, list the vertices and the corresponding Pythagorean equation in the chart below. An example has been done for you.*

Triangle	Pythagorean Equation
ABD	$a^2 + h^2 = d^2$

Problem 3.8 *1) Using the three Pythagorean equations you found in the previous exercise, and the comparison between h and r in problem 3.6, solve for h in terms of a and b. Conclude that $GM(a,b) \leq AM(a,b)$.*

2)When does equality hold? (Think geometrically!) Why is it implied that this argument only works for $a, b \geq 0$.

We now have two proofs that if a and b are positive, then $\sqrt{ab} \leq \frac{a+b}{2}$. The next step is to show that for all $n \geq 2$,

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Let's look back at the first table you filled in at the beginning of this packet. You found that

(a_1, a_2)	$GM(a_1, a_2)$	$AM(a_1, a_2)$
(1, 9)	3	5
(2,8)	4	5
(5,5)	5	5

Notice that in the above table, as you go down the rows, a_1 is increased by the same amount that a_2 is decreased. When that happens the geometric mean goes up while the arithmetic mean remains the same. Let's see if we can formulate this observation as a general property.

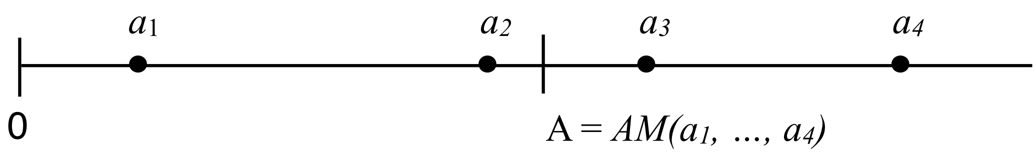
Problem 3.9 *Let* $0 < x < y$ *and let* $k > 0$ *satisfy that* $x + k \leq y - k$*. Show that* $AM(x, y) = AM(x + k, y - k)$ *and* $xy < (x + k)(y - k)$*. Conclude that* $GM(x, y) < GM(x + k, y - k)$*.*

We will now prove a simplified version of the AM-GM inequality. Suppose we have four positive numbers a_1, a_2, a_3, and a_4 with the following four properties:

Properties:

1. $0 < a_1 < a_2 < a_3 < a_4$
2. $A = AM(a_1, a_2, a_3, a_4)$
3. $a_2 < A < a_3$
4. the distance from a_2 to A is smaller than the distance from A to a_3 (i.e. $A - a_2 < a_3 - A$)

Pictorially, we have the following diagram.



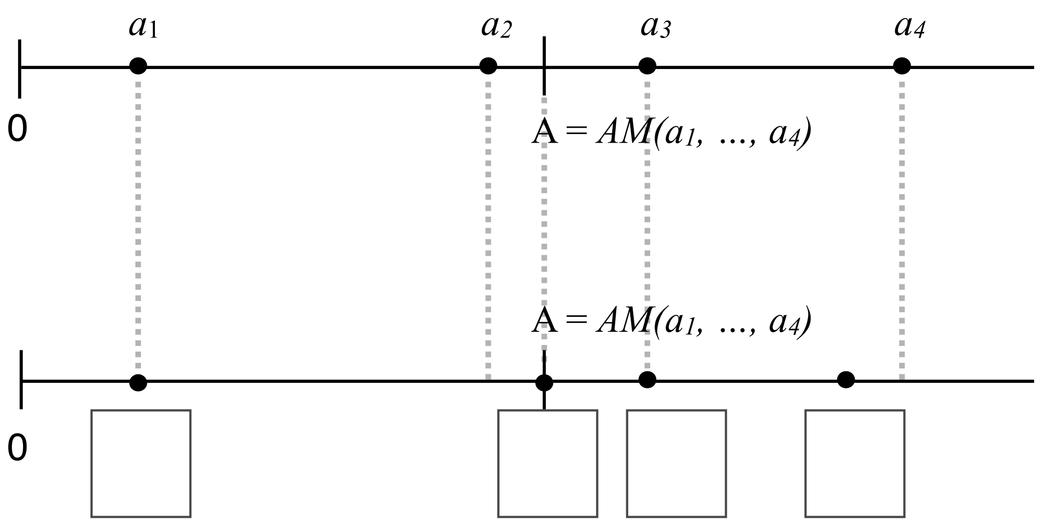
Let's show that $GM(a_1, ..., a_4) \leq AM(a_1, ..., a_4) = A$.

Problem 3.10 *Let $k = A - a_2$. Solve for A then use that to show that $a_2 + k < a_3 - k$? (Hint: Try to use property 3 and the number line to give a geometric argument for why you expect this inequality to be true. Then prove the result algebraically using the inequality in property 3.)*

Problem 3.11 *Why is $a_2 + k < a_4 - k$?*

Now let's consider four positive numbers b_1, b_2, b_3 and b_4 with $b_1 = a_1$, $b_2 = a_2 + k$, $b_3 = a_3$ and $b_4 = a_4 - k$.

Problem 3.12 *On the next page is the number line for a_1, a_2, a_3 and a_4 shown earlier and a new number line. In the new number line, fill in the empty boxes with b_1, b_2, b_3, b_4 in the order that they should appear.*

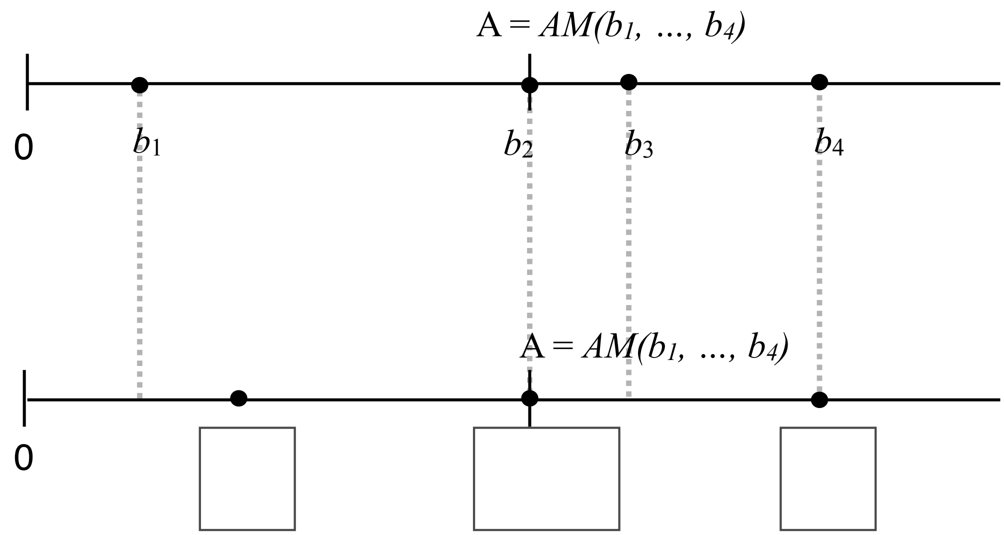


Problem 3.13 *Show that* $A = AM(a_1, ..., a_4) = AM(b_1, ..., b_4)$. *Why is* $GM(a_1, ..., a_4) < GM(b_1, ..., b_4)$*? (Hint: Reread Problem 3.9). Which of the numbers* b_1, b_2, b_3, b_4 *are equal to* A*?*

From the new number line you constructed in Problem 3.12 we see that $b_2 = A$ and b_3 is the closest number to A that is not already equal to A.

Let $k_2 = b_3 - A$ and define c_1, c_2, c_3, c_4 by $c_1 = b_1 + k_2$, $c_2 = b_2, c_3 = b_3 - k_2$ and $c_4 = b_4$.

Problem 3.14 *Two number lines are shown below. In the second number line, fill in the empty boxes with c_1, c_2, c_3 and c_4 in the order that they should appear. You may need to put multiple numbers in the same box.*

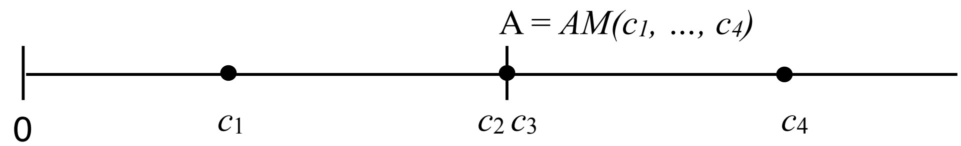


Problem 3.15 *Why is $AM(c_1, ..., c_4) = AM(b_1, ..., b_4) = A$? Is $GM(b_1, ..., b_4) < GM(c_1, ..., c_4)$? Which of the numbers c_1, c_2, c_3, c_4 are equal to A?*

To recap we turned the numbers $a_1, ..., a_4$ into $b_1, ..., b_4$ and then into $c_1, ..., c_4$:

$$a_1, a_2, a_3, a_4 \longrightarrow b_1, b_2, b_3, b_4 \longrightarrow c_1, c_2, c_3, c_4$$

Let's turn $c_1, ..., c_4$ into a new set of numbers. In Problem 3.14 you constructed the following number line.



Problem 3.16 *Why is* $A - c_1 = c_4 - A$*? Let* $d_1 = d_2 = d_3 = d_4 = A$*. Why is* $AM(d_1, ..., d_4) = A$ *and* $GM(c_1, ..., c_4) < GM(d_1, ..., d_4)$*?*

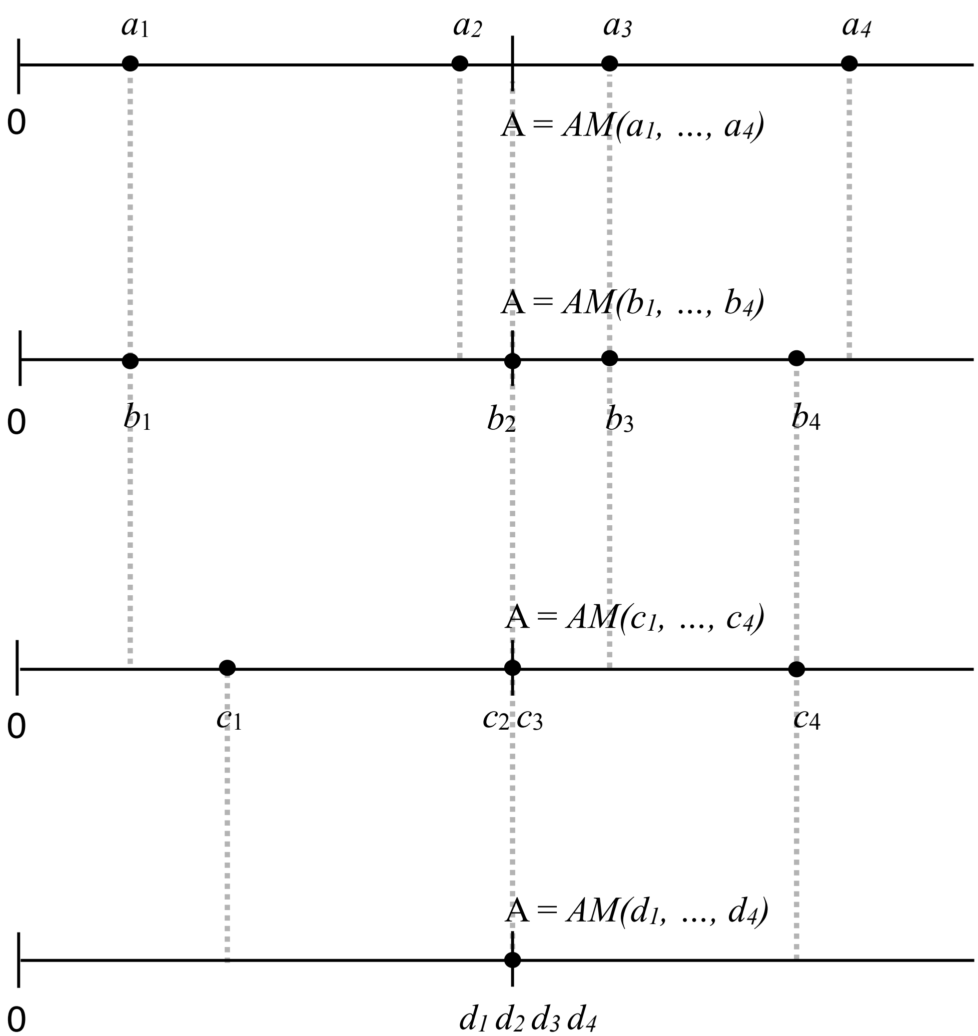
To recap we have the four number lines shown on the following page. You also showed that

$$AM(a_1, ..., a_4) = AM(d_1, ..., d_4) = A$$

and

$$GM(a_1, ..., a_4) < GM(b_1, ..., b_4) < GM(c_1, ..., c_4) < GM(d_1, ..., d_4).$$

Problem 3.17 *Why is* $GM(a_1, ..., a_4) < AM(a_1, ..., a_4)$*?*



$$A = AM(a_1, \ldots, a_4) = AM(b_1, \ldots, b_4) = AM(c_1, \ldots, c_4) = AM(d_1, \ldots, d_4)$$

$$GM(a_1, \ldots, a_4) < GM(b_1, \ldots, b_4) < GM(c_1, \ldots, c_4) < GM(d_1, \ldots, d_4)$$

Problem 3.18 *We originally stipulated that $a_1 < a_2 < a_3 < a_4$. How can we change the inequalities so that $GM(a_1, ..., a_4) = AM(a_1, ..., a_4)$?*

Congraulations. You have now worked through a simplified version of the AM-GM inequality. The proof of the full version of the AM-GM inequality is similar to the one you worked through, but requires an advanced mathematical technique called induction. As a reminder, the full version of the AM-GM inequality is

AM-GM Inequality

If $a_1, ...a_n$ are positive numbers, then

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}$$

and equality holds if and only if $a_1 = \cdots = a_n$.

4 Challenge Problems

Suppose a box has surface area A. What dimensions should the box have so that it has the maximum possible volume? You might instinctively think that the box should be a cube, and using the AM-GM inequality, we can show that this is correct.

Problem 4.1 *Suppose a box with surface area A has dimensions a, b, and c.*

1. *Write down a formula for A in terms of a,b, c.*
2. *What is the volume of a cube with surface area A. Write your answer in terms of A.*
3. *Show that $abc \leq (\frac{A}{6})^{3/2}$. Why does this show that the box with surface area A that has maximum volume is a cube?*

Problem 4.2 *Let a, b and c be positive numbers. Show that $a+b+c \leq \frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}$. (2002 Canadian Math Olympiad).*

Problem 4.3 *Let* $p_1, ..., p_n$ *be rational numbers between 0 and 1 with* $p_1 + \cdots + p_n = 1$. *Use the AM-GM inequality to show that* $a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n$. *(Hint: Start by letting* $p_1 = {}^{d_1}/_{f_1}, ..., {}^{d_n}/_{f_n}$ *where the fractions are in simplified form.)*