Math Circle Monday March 20, 2017

Calculating Cardinalities

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To say that **a set** A **has** 5 **elements** means that we can write the elements of A as a list

$$a_1, a_2, a_3, a_4, a_5$$

in a way that

(a) There are no repetitions, and
(b) Every element of A appears on the list.

More formally, this says that we can construct an assignment (that is, a map) from the set $\{1, 2, 3, 4, 5\}$ to A in such a way that

(a) To distinct numbers (from 1 to 5) we assign distinct elements of A, and
(b) **Every** element of A is assigned to some number.

For instance, the set $A = \{\sharp, \flat, \square, \heartsuit, \spadesuit\}$ has 5 elements because we can write down an assignment/list with the desired properties, for example:

$$\begin{array}{ccc} 1 & \mapsto & \sharp \\ 2 & \mapsto & \flat \\ 3 & \mapsto & \square \\ 4 & \mapsto & \heartsuit \\ 5 & \mapsto & \spadesuit \end{array} \qquad \begin{array}{c} a_1 = \sharp \\ a_2 = \flat \\ a_3 = \square \\ a_4 = \heartsuit \\ a_5 = \spadesuit \end{array}$$

Task 1. How many assignments/lists can you create?

Task 2. If we tried to construct a similar assignment between the set $\{1, 2, 3, 4\}$ and $A = \{\sharp, \flat, \square, \heartsuit, \spadesuit\}$, which of the properties mentioned above would fail?

Task 3. If we tried to construct a similar assignment between the set $\{1, 2, 3, 4, 5, 6\}$ and $A = \{\sharp, \flat, \square, \heartsuit, \spadesuit\}$, which of the properties mentioned above would fail?

Similar considerations apply if we replace number n with any positive integer n.

DEFINITION. An assignment between two sets is called **bijective** precisely when requirements (a),(b) are satisfied. Similarly we define a bijective list.

DEFINITION. Two sets are called **equinumeros**, which means they have the "same number of elements" preciesly when there is a bijective assignment between them. We also say the two sets have the same **cardinality**.

Thus **cardinality** is the number of elements of a set: a set A has cardinality n precisely when we can construct a bijection between the set $\{1, 2, \ldots, n\}$ and A. Equivalently, A has cardinality n precisely when we can write down a non-repetitive list $a_1, a_2, \ldots, a_n$ which covers the entire set A.

Now this idea can be applied to infinite sets as well. The basic sample set is the set of all non-negative integers $\mathbb{N}$. Thus,

$$\mathbb{N} = \{0, 1, 2, 3, \ldots, n, \ldots\}$$

DEFINITION Sets which have the "same number of elements as $\mathbb{N}$", that is, which are equinumeros with $\mathbb{N}$, are called **countable**. All other infinite sets are called **uncountable**.

Thus, a set A is countable iff we can write a non-repetitive list

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

indexed by non-negative integers which covers the entire set A. For example, the set of all positive integers is countable, because we can list the elements of this set as follows:

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$
1	2	3	4	$\ldots$	$n+1$	$\ldots$

Here $a_n = n + 1$. Notice the shift of index needed in the formula.

A. COUNTABLE SETS

In the following we will explore which infinite sets are countable. In each case complete the table to visualizes the assignment/list and then write down a formula.

Task 4. The set of all even non-negative integers is countable.

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

Similarly, the set of all non-negative integers divisible by 5 (or by any other fixed positive integer like 3 or 17) is countable.

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

Note: *these are proper subsets of* $\mathbb{N}$*, yet they all have the same cardinality as* $\mathbb{N}$.

Task 5. If B is a countable set and we add a new element c to B, the resulting set A will be countable.

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

Task 6. If B, C are countable sets then the union $A = B \cup C$ (containing all elements that are in B *or* in C) is countable. Hint: start from a listing of the elements of B and C, then find a consistent way to list **all** the elements of $A = B \cup C$ **with no repetition**.

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

What is the formula for a_n?

Similarly any finite union of countable sets is countable.

Task 7. The set of all integers $\mathbb{Z}$ is countable.

a_0	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

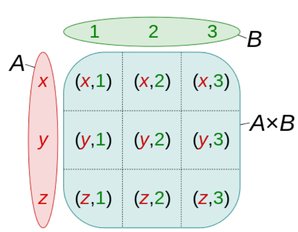
What is the formula for a_n?

Task 8. The set of all primes is countable. In this case, providing a formula for a_n would be hard. Can you find another convincing argument?

More generally, any infinite subset of $\mathbb{N}$ is countable.

DEFINITION The Cartesian product of two sets A and B is the set of all **ordered** pairs (a, b) such that $a \in A$ and $b \in B$:

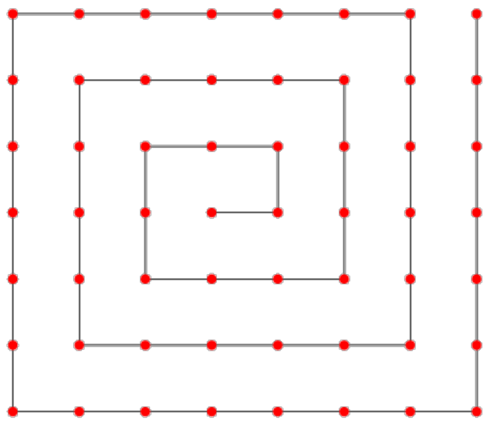
$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$



The elements of a cartesian product of two countable sets can be arranged in a lattice. Instead of explicitly listing all the elements of the lattice, we can draw a path through the lattice such that

(a) every point of the lattice is hit by the path
(b) no point of the lattice is hit more than once.

An example of such path is the spiral pictured below:



Task 9. The sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ are countable. Hint: Draw the elements in a lattice; label all the points in the lattice, than draw a path with the properties mentioned above (be creative!).

More generally, if A and B are countable sets, then the Cartesian product $A \times B$ is countable.

Task 10. The set $\mathbb{Q}$ of all rational numbers is countable. (Hint: Can you represent points of $\mathbb{Q}$ as points of a lattice? In order to make sure you are not double listing any element, you may need to erase some points of the lattice....)

Task 11. If A is infinite and there is a list

$$a_0, a_1, \ldots, a_n, \ldots,$$

which covers all of A, then there is such a list which is non-repetitive. This meants that A is countable.

Task 12. If $A_0, A_1, A_2, \ldots$ is a countable list of countable sets then the union

$$A = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n \ldots$$

is countable.

B. SETS IN A PLANE

Task 13. Two bounded lines in plane without endpoints are equinumeros.

Task 14. A bounded line in plane without endpoints is equinumeros with an arc without endpoints.

Task 15. An arc in plane without endpoints is equinumeros with an unbounded line.

Task 16. A bounded line in plane without endpoints is equinumeros with an unbounded line.

Task 17. A bounded line in plane without endpoints is equinumeros with a bounded line with one endpoint and also with bounded line with both endpoints.

Task 18. Any two circles in plane are equinumeros. Any two empty squares in plane are equinumeros.

Task 19. Any two discs in plane are equinumeros. Any two full squares in plane are equinumeros.

Task 20. Any circle in plane is equinumeros with any empty square. Any disc is equinumeros to any full square.

Task 21. A full square without one corner point is equinumeros with the full square with all corner points.

Task 22. A bounded line with left endpoint in plane is equinumeros with a full square without one corner point. *Sample case*: The half-open interval $[0, 1)$ is equinumeros with the square

$$[0, 1) \times [0, 1)$$

in plane, that is, with a full square without the upper right corner point $(1, 1)$.

C. UNCOUNTABLE SETS

Task 23. Open interval $(0,1)$ is uncountable. (So we have another infinite cardinality!) It also means that the set of all real numbers $\mathbb{R}$ is uncountable and all sets in Section B are uncountable.

Task 24. If A is any set then there does not exist any assignment

$$A \to \mathcal{P}(A)$$

which covers the entire set $\mathcal{P}(A)$. Here $\mathcal{P}(A)$ is the **power set** of A. It consists of all subsets of A. Symbolically:

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}$$