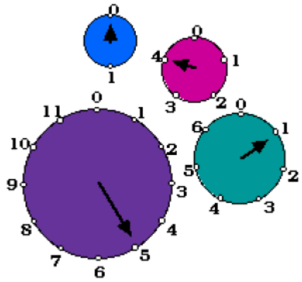
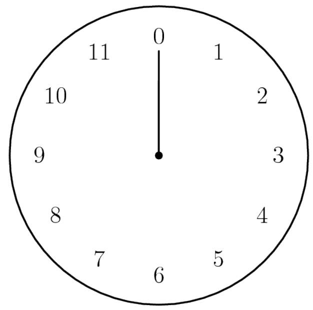
Clock Arithmetic



1. Pretend that it is 3:00 now (ignore am/pm).
   (a) What time will it be in 17 hours? ____________

   (b) What time was it 22 hours ago? ____________

   (c) The clock on the right has a 0 in place of the 12. How can you use this clock to answer the questions above?



2. In the "clock" on the right, we can label Sunday as 0, Monday as 1, . . . , Saturday as 6.
   (a) What day was it 10 days ago? __________________

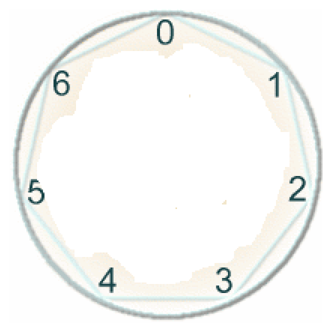
   (b) What day will it be in 15 days? __________________

   (c) What day was it on this date last year (366 days ago)?

   ____

   (d) What day will it be on this date next year (365 days from now)?

   ____



3. The chip game
   (a) Play the following game with a partner. Start with 7 chips in a pile. You and your partner take turns removing chips; you have the choice of removing 1 or 2 chips on each turn. The last one to remove a chip loses.
   Take turns being the first player. Is there a strategy that ensures that one of the players wins?

   (b) Now try the game starting with 8 chips. Is there a strategy that ensures that one of the players wins?

(c) Now try with 9 chips. Is there a strategy?

Clock arithmetic

4. Let's write $m \pmod{n} = b$ when $m \div n$ has remainder b.
   (a) $20 \pmod{12} =$ ____________
   (b) $3 + 5 \pmod{12} =$ ____________
   (c) $12 + 11 \pmod{7} =$ ____________
   (d) $12 \pmod{7} + 11 \pmod{7} =$ ____________
   (e) $(12 \pmod{7} + 11 \pmod{7}) \pmod{7} =$ ____________
   (f) Compare your answers in (c) and (e). What shortcut does this give you in clock arithmetic?
   (g) $771 + 75 \pmod{7} =$ ____________
5. Multiplication in clock arithmetic
   (a) $5 \times 5 \pmod{12} =$ ____________
   (b) $5 \times 6 \pmod{7} =$ ____________
   (c) $5 \pmod{4} \times 5 \pmod{4} =$ ____________
   (d) $5 \times 5 \pmod{4} =$ ____________
   (e) Compare your answers to the last two parts. Notice a similar shortcut like above; now try the following:
   $445 \times 43 \pmod{4} =$ ____________
6. **Odd and even**
   Build an addition and multiplication table $\pmod{2}$:

+	0	1
0		
1		

×	0	1
0		
1		

(a) If a number a is odd, then what is $a \pmod 2$? ____________

(b) $286 \times 1034 \pmod 2 =$ ____________

(c) Suppose the product of two numbers $a \times b$ is odd. Using the multiplication table, what can you conclude about the numbers a and b?

7. Divisibility by 3

(a) Consider the number 123. Add its digits: $1 + 2 + 3 = 6$, which is divisible by 3; and 123 is divisible by 3 also. Take another 3-digit number that is not divisible by 3. Add its digits together; is the sum divisible by 3?

(b) Try a couple of 2-digit numbers and 4-digit numbers. What can you say when the sum of the digits is divisible by 3?

(c) Remember the chip game? Notice that $7 \pmod 3 = 1$. How does this affect the winning strategy when starting with 7 chips? How about when starting with 8 or 9 chips?

(d) The 5-digit numbers $12AB3$ and $23ABC$ are each divisible by 3. (A, B, C are single digits.) Suppose that C is an even number. What can you conclude about C?

8. A trip to the candy store.

(a) Sally buys 2 candy bars. They both have the same price. The cashier charges Sally \$1.99. Did the cashier make a mistake?

(b) Harry buys:

11 snickers bars,

19 peanut butter cups,

3 kit kat bars.

The snickers bars and the peanut butter cups have the same price. The cashier charges Harry \$18.35. Harry says that the cashier charged him the wrong price. How does Harry know?

9. Hot dog!

   (a) I am buying hot dogs and buns. My store has hot dog buns in packages of 26, and hot dogs in packages of 8. What is the smallest number of packages of hot dog buns and hot dogs that I can buy if I want the same number of hot dogs and buns?

   (b) My store gets a new brand of hot dog buns that come in packages of 15. Now what is the smallest number of packages of hot dog buns and hot dogs that I can buy if I want the same number of hot dogs and buns? Do I end up with fewer hot dogs this time?

10. The $\pmod{n}$ Game

    The $\pmod 5$ game is as follows: write the numbers $1, 2, 3, 4$ in a list. The running total starts at 0. Two players take turns; on your turn, you select a number from the list, add it $\pmod 5$ to the running total, and cross it off the list. The first player to reach 0 loses; if the running total never reaches 0 when all the numbers are crossed out, the game ends in a tie.

    (a) Example: Alice selects 3 and crosses it off. The running total is 3. John selects 2 and crosses it off. The running total is $3 + 2 \pmod 5 = 0$, so Alice wins.

(b) Take turns being the first player of the game. Is there a winning strategy?

(c) Now try the (mod 6) game: write the numbers $1, 2, 3, 4, 5$ and play as before, but each time you add (mod 6) to the running total. Is there a winning strategy?

11. **Powerful powers**

Recall that $a^2 = a \times a,\ a^3 = a \times a \times a,\ \ldots$.

(a) What is $2^1 \pmod 7$? ____________

(b) How about $2^2 \pmod 7$? ____________ $2^3 \pmod 7$? ____________

(c) Can there be a whole number n with $2^n \pmod 7 = 5$? ____________

(d) What is $2^{333} \pmod 7$? ____________

12. **Coconuts**

Four people are shipwrecked and there are only coconuts to eat. They collect all the coconuts they can find and weary from their work fall asleep.

In the night one of the castaways wakes up and secretly divides the coconuts into four equal piles, she hides her share and throws to the monkeys the three that were left over before putting all the remaining nuts back into one pile.

Later another of the castaways wakes up and she too secretly divides the coconuts into four equal piles, she hides her share and throws to the monkeys the three that were left over before putting all the remaining nuts back into one pile.

Later still, yet another of the castaways wakes up and she too secretly divides the coconuts into four equal piles, hides her share and throws to the monkeys the three that were left over before putting all the remaining nuts back into one pile.

Just before morning the last castaway wakes up and she too secretly divides the coconuts into four equal piles, hides her share and throws to the monkeys the three that were left over before putting all the remaining nuts back into one pile.

Next morning with a much reduced pile the four castaways find they can share out equally all the coconuts that are left!

What is the least number of coconuts they could have started with?

13. **Odds and ends.**

(a) $22^{55} - 55^{22} \pmod 7 =$

(b) What is the ones digit of

$$1^2 + 2^2 + 3^2 + \cdots + 97^2 + 98^2 + 99^2 + 100^2?$$

(c) The nth term of a sequence is given by the formula $n^3 + 11n$. Find the first 4 terms of the sequence. Show that all the terms of the sequence are divisible by 6.