WHAT IS A LINE?

Abstract. You 'know' what a straight line is, but why? What makes it 'straight?' Below we will explore situations where this notion is quite a bit different.

Teaching Plan

I envision a very brief discussion recalling basic aspects of Euclidean geometry as in section 1. Then a brief introduction to Taxicab geometry followed by the students breaking into groups to answer questions. We discuss the answers as a group and then proceed to introduce dogs and frisbees. We return to discussion groups to answer questions. We zoom out and discuss answers as a group. Ideally this will take approximately 45 minutes, leaving 45 minutes for a collective discussion lead by the lecturer on placing a geometry on the space of triangles, hopefully with the students taking the lead in suggesting ideas.

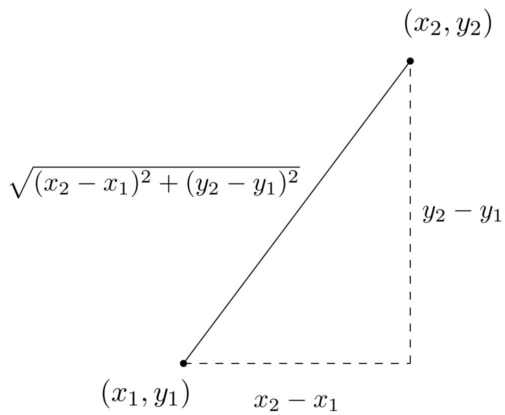
Date: October 17, 2016.

1. Euclidean Geometry

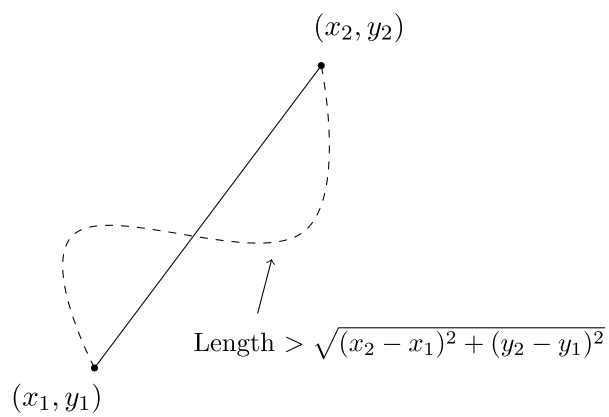
Let us begin by recalling some basic facts about Euclidean geometry. First, given two points (x_1, y_1), (x_2, y_2), we 'know' that the distance between them is given by

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is related to the Pythagorean theorem, as seen in this figure:



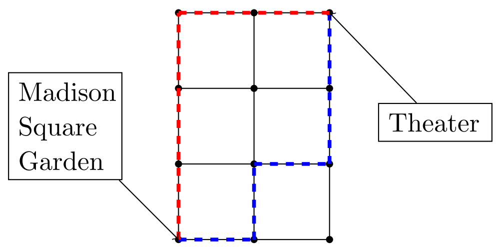
This idea comes packaged with the fact that, not only does this represent the 'distance,' but this distance is achieved as the length of the line segment connecting them. In particular, the length of the segment is shorter than the length of any other curve connecting the two points:



In this session we will explore more general geometric situations where, depending on the situation, the meaning of "straight line" can be quite different, and that we can construct "geometries" in all kinds of interesting situations.

2. Taxicab Geometry

A cab driver in New York picks up a passenger at Madison Square Garden and asks to travel to a theater which is four blocks north and two blocks east. He is forced to follow the roads, which are laid out on a grid. Note that he will have more than one option for how to travel. Two paths are shown below:



1. What is the length of the shortest path he can take? How many different paths can you give which have that length?

2. Express each path you found above in as a series of directions, writing N for moving north, E for moving east. In particular, the red path above is NNNEE, the blue path is ENENN.

3. Now suppose on his next trip the cab driver picks up someone at Madison Square Garden and is asked to take a visitor X blocks east and Y blocks north. What is the shortest length of a path he can take? How many such shortest paths are there? (**Hint:** Think about the process of writing down all paths as in the previous exercise? How many such ways are there?)

4. Now the cab driver is tasked with driving 5 blocks east and 8 blocks north. However, the intersection 2 blocks east and 3 blocks north is blocked by construction. Now how many paths are available to the driver?

3. Dogs and frisbees

Picture yourself at the beach. Your trusty dog Checkers is with you, and you are about to play your favorite game: you throw the frisbee down the beach and into the water, and Checkers goes to retrieve it. On your first throw, your are standing at the water's edge. you throw it 30 meters down the beach, and 5 meters into the water. Checkers is a very competitive dog, and wants to get the frisbee as fast as possible. He runs down the beach at 2 meters per second, and he swims at 1 meter per second.

1. Suppose Checkers swims straight to the frisbee. How long does it take him?

2. Suppose Checkers runs down the beach until he is directly across from the frisbee then swims straight out to it. How long does it take him?

3. Suppose Checkers runs 10 meters down the beach then swims directly towards the frisbee. How long does it take him?

4. Suppose Checkers runs B meters down the beach then swims directly towards the frisbee. Write a formula for how long it takes him.

If we had the tools of **calculus**, we could find the value B that minimizes the time, which would then give us the required path. Let's modify the problem to one which we can solve without these tools. Let's say that instead of minimizing the overall time, Checkers decides to minimize a different function of time. In particular, let t_B denote the time he spends on the beach and t_O denote the time he spends in the ocean, and suppose Checkers wants to minimize $t_B^2 + t_O^2$.

5. Find the path that minimizes $t_B^2 + t_O^2$.

4. The space of triangles

What these prior examples hopeffully suggest to you is that the idea of 'distance,' and 'straight' is extremely flexible, and is really derived from whatever situation is at hand. Let's go further down the rabbit hole and ask, **is there a 'distance' on the space of triangles?'** To begin, lets make clear that we do NOT mean the distance between two triangles as they sit in particular in some plane. We want to ask what the distance is between two *shapes of triangles*.

First of all, we want to consider all congruent triangles as the *same triangle*. Moreover, we want to quantitatively capture that these two triangles, irrespective of their positioning in space, are 'close'. In principle there are many answers to this. One natural way is to take the absolute value of the difference between pairs of angles, or between pairs of sides. One also has to think about a lot of issues: do we want to fix the area? Or fix the length of one side to be 1? What about similar questions for parallelograms?

5. Bonus Problems

(1) Our cabbie is asked to go m blocks east and n blocks north. There are two potholes, one at (k_1, l_1), the other at (k_2, l_2). How many paths are available to our cabbie? How many different types of paths there are, depending on which side of the potholes we pass on?

(2) Can you construct a beach and an ocean where Checkers' path takes the form of a trapezoid? How about a triangle?