# Tic-Tac-Toe and its Variants

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1   What is Tic-Tac-Toe?

Tic-Tac-Toe is one of the simplest games you can play with pen and paper. Here are the rules:

Let's say you mark X and your friend marks O. Players take turns marking empty boxes in a 3×3 grid. To win, you must get your three marks in a row (vertically, horizontally, or diagonally) before your friend does. If nobody gets three in a row and all boxes are full, then the game ends in a draw.

1. Play a few rounds of Tic-Tac-Toe with the person sitting next to you.
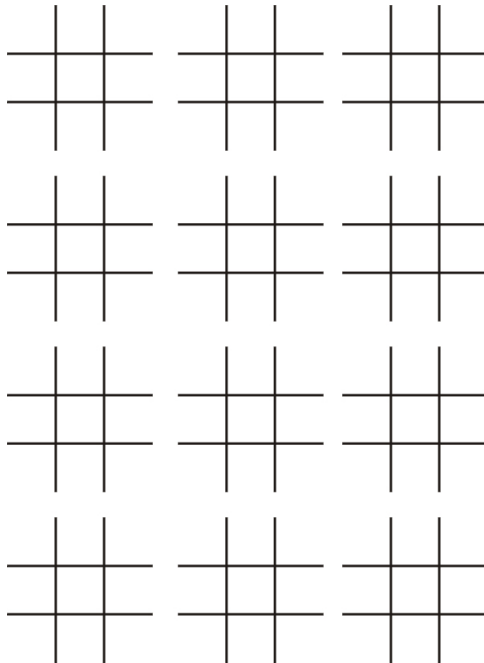
Some Counting:

2. It's easy to check that there are $3 \times 3 = 9$ total squares. How many lines are there on a Tic-Tac-Toe board?

3. How many of these lines pass through the center square?

4. Choose a corner square. How many lines pass through it?

5. Choose a square that is not the center or a corner. How many lines pass through it?

6. If you play a bunch of times, you will notice that almost every game ends in a draw. Describe a strategy for the second player to follow so that the first player cannot force a win.

2 3 × 3 × 3 Tic-Tac-Toe

Now let's play Tic-Tac-Toe in three dimensions. We represent the $3 \times 3 \times 3$ board by stacking three regular Tic-Tac-Toe boards on top of each other (low, medium, and high). The goal is the same as in 3×3 Tic-Tac-Toe; get three marks in a row (horizontally, vertically, or diagonally). Here are some examples of lines in this game.
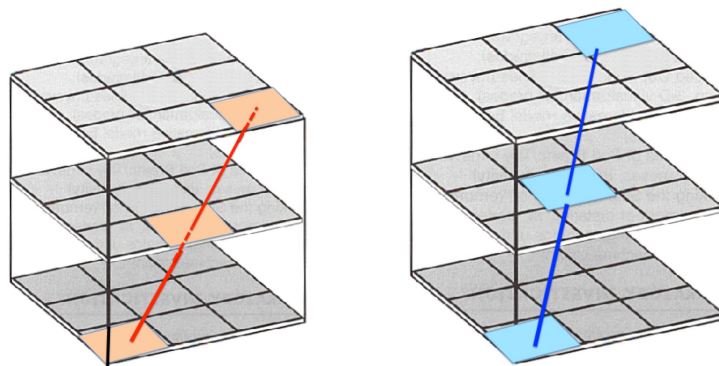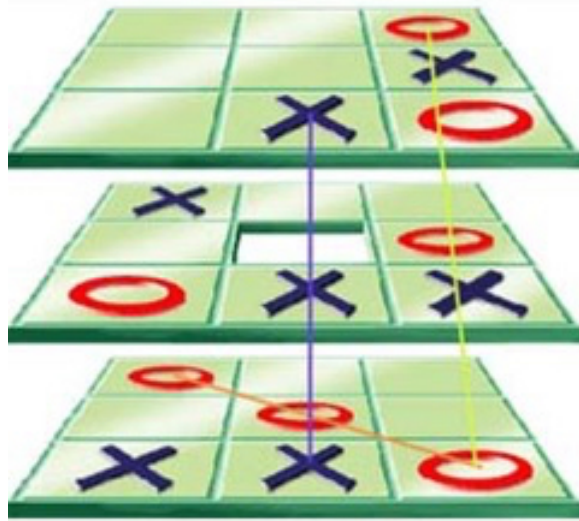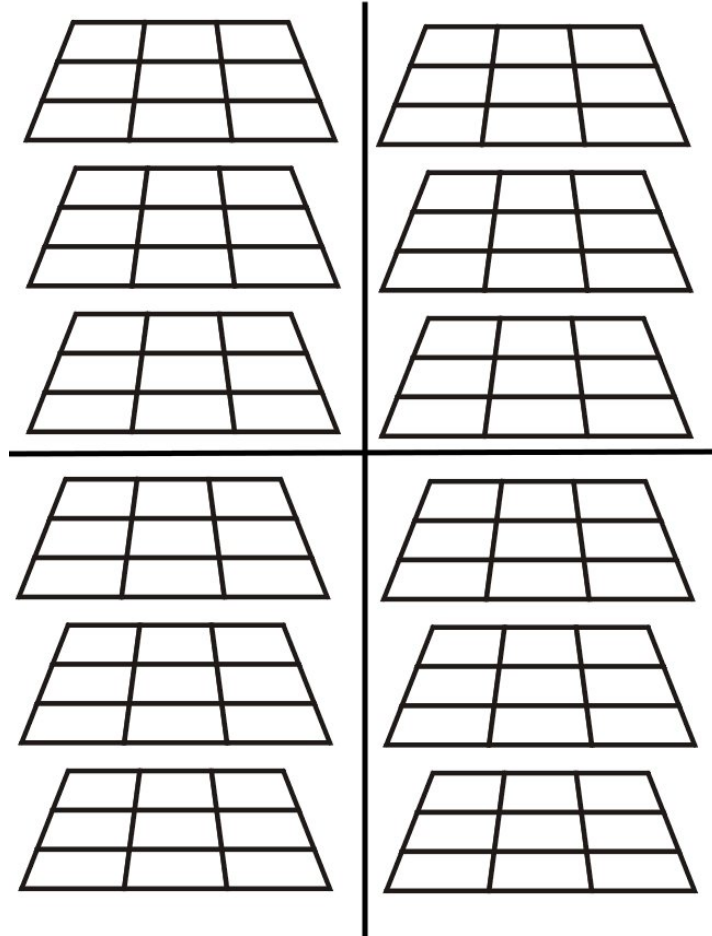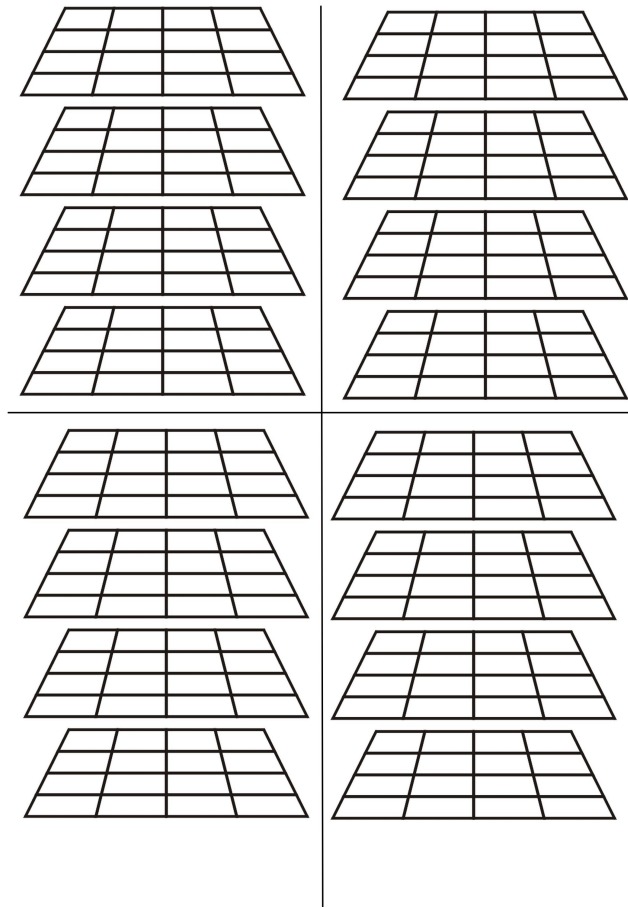
1. Play $3 \times 3 \times 3$ Tic-Tac-Toe a few rounds. Who usually wins?

Some Counting

2. How many squares are on a $3 \times 3 \times 3$ Tic-Tac-Toe board?

3. How many lines are there on a $3 \times 3 \times 3$ Tic-Tac-Toe board?

4. How many of these lines pass through the center?

5. How many pass through each other square?

6. Consider a variation of the game where the first player is allowed to go anywhere on the first move, except in the center, and the rest of the game is exactly the same as before. Who wins in this version of the game? Can you prove it?

7. Is it possible for the game to end in a draw?

3 Playing Tic-Tac-Toe in a larger board

1. Play $4 \times 4 \times 4$ Tic-Tac-Toe a few rounds. Which player usually wins? Can you show that this player has a winning strategy?

Some Counting

2. How many squares are on a $4 \times 4 \times 4$ Tic-Tac-Toe board?

3. How many lines are there on a $4 \times 4 \times 4$ Tic-Tac-Toe board?

4. How many of these lines pass through the center?

5. How many pass through each other square?

$[n]^k$ **Tic-Tac-Toe**

We can play a version of Tic-Tac-Toe on a board of any size. We need only specify how many squares are in each row (we call this n) and how many different 'dimensions' there are (we call this k). The game is played just like regular Tic-Tac-Toe, and it takes n square in a row (horizontally, vertically, or diagonally) to win. So the original Tic-Tac-Toe has $n = 3$ and $k = 2$. The 3D version described on the previous page has $n = k = 3$, and the $4 \times 4 \times 4$ game has $n = 4$ and $k = 3$. We can ask questions about the $[n]^k$ Tic-Tac-Toe games.

1. How many squares are there on an $n \times n \times \cdots \times n$ Tic-Tac-Toe board where there are k different n's? That is, how many square are there in the $[n]^k$ game?

2. What do the lines look like in the $[n]^k$ game? How many different lines are there?

3. Use the fact that the 3×3 Tic-Tac-Toe game is a draw to show that the 5×5 Tic-Tac-Toe game is a draw.

4. Use the same idea to prove the following statement: If the $[n]^k$ Tic-Tac-Toe game is a draw, then the $[n+2]^k$ Tic-Tac-Toe game is a draw.

5. In every version of Tic-Tac-Toe we have seen, the game is either a win for the first player, or a draw. Show that this is true for the $[n]^k$ Tic-Tac-Toe game for any n and k. That is, why can't the second player ever have an advantage that allows her to force a victory?

Unsolved Problems:

1. Is the $5 \times 5 \times 5$ Tic-Tac-Toe game (the $[5]^3$ game) a win for the first player or a draw?

2. Is the $5 \times 5 \times 5 \times 5$ Tic-Tac-Toe game (the $[5]^4$ game) a win for the first player or a draw?

3. Suppose that the $[n]^k$ Tic-Tac-Toe game is a draw. Show that the $[n+1]^k$ game is a draw.