FUNCTIONAL EQUATIONS

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1. What is a functional equation

An equation contains an unknown function is called a functional equation.

Example 1.1 The following equations can be regarded as functional equations

$$\begin{aligned} &f(x) = -f(-x), && \text{odd function}\\ &f(x) = f(-x), && \text{even function}\\ &f(x+a) = f(x), && \text{periodic function, if } a \neq 0 \end{aligned}$$

Example 1.2 The *Fibonacci* sequence

$$a_{n+1} = a_n + a_{n-1}$$

defines a functional equation with the domain of which being nonnegative integers. We can also represent the sequence is

$$f(n+1) = f(n) + f(n-1).$$

Example 1.3 (Radioactive decay) Let $f(x)$ represent a measurement of the number of a specific type of radioactive nuclei in a sample of material at a given time x. We assume that initially, there is 1 gram of the sample, that is, $f(0) = 1$. By the physical law, we have

$$f(x)f(y) = f(x+y).$$

Can we determine which function this is?

2. Substitution method

Example 2.1 Let $a \neq 1$. Solve the equation

$$af(x) + f(\frac{1}{x}) = ax,$$

where the domain of f is the set of all non-zero real numbers.

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Solution: Replacing x by x^{-1}, we get

$$af(\frac{1}{x}) + f(x) = \frac{a}{x}.$$

We therefore have

$$(a^2 - 1)f(x) = a^2 x - \frac{a}{x},$$

and hence

$$f(x) = \frac{a^2 x - \frac{a}{x}}{a^2 - 1}.$$

$\square$

Exercise 2.2 Solving the functional equation ($a^2 \neq b^2$)

$$af(x-1) + bf(1-x) = cx.$$

Exercise 2.3 Finding a function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that

$$(1 + f(x^{-1}))(f(x) - (f(x))^{-1}) = \frac{(x-a)(1-ax)}{x},$$

where $a \in (0, 1)$.

3. Recurrence Relations

Example 3.1 (Fibonacci Equations) Let

$$f(n+2) = f(n+1) + f(n)$$

with $f(0) = 0, f(1) = 1$. Find a general formula for the sequence.

Solution: We consider the solution of the form

$$f(n) = \beta^n$$

for some real number β. Then we have

$$\beta^{n+2} = \beta^{n+1} + \beta^n$$

from which we conclude that $\beta^2 = \beta + 1$. Therefore

$$\beta_1 = \frac{1+\sqrt{5}}{2}, \quad \beta_2 = \frac{1-\sqrt{5}}{2}.$$

A general solution of the sequence can be written as

$$f(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where c_1, c_2 are coefficients determined by the initial values. By the initial conditions, we have

$$\begin{aligned} &c_1 + c_2 = 0 \\ &c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}$$

Thus we have

$$c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}.$$

Thus

$$f(n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

It is interesting to see that the above expression provide all positive integers for any n.

□

Exercise 3.2 Solving the sequence defined by

$$a_n = 3a_{n-1} - 2a_{n-2}$$

for $n \geq 2$ with the initial condition $a_0 = 0, a_1 = 1$.

4. The Cauchy's Method

Example 4.1 Assume that f is a continuous function on $\mathbb{R}$. Assume that for any $x, y \in \mathbb{R}$

$$f(x+y) = f(x) + f(y).$$

Find the function $f(x)$.

Solution: First, we have $f(0) = 0$. Let $c = f(1)$.

Using the math induction, we have

$$f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n).$$

Let $x_1 = \cdots = x_n = x$. Then we get

$$f(nx) = nf(x)$$

for any positive integer n. Let $x = 1/m$ where m is a nonzero integer. Then we have

$$f(\frac{n}{m}) = nf(\frac{1}{m}).$$

On the other hand,

$$mf(\frac{1}{m}) = f(1) = c.$$

Thus we have

$$f(\frac{n}{m}) = nf(\frac{1}{m}) = c\frac{n}{m}.$$

The conclusion here is that for any *rational* number γ, we have

$$f(\gamma) = c\gamma.$$

If f is continuous, then we conclude that for any real number x,

$$f(x) = cx = x \cdot f(1).$$

For those of you who are not familiar with the concept of continuity, the assumption can be weakened to the boundedness of the function. Assume that f is bounded. Let x be any real number. For any $\epsilon > 0$, we choose a rational number γ such that $|x - \gamma| < \epsilon$. Let N be the integer part of the $1/\sqrt{\epsilon}$. Then

$$|f(N(x-\gamma))| \leq C$$

because the function is bounded. Thus we have

$$|f(x) - f(1)\gamma| = |f(x) - f(\gamma)| \leq \frac{C}{N}.$$

If we choose ϵ to be so small, then we must have

$$f(x) = f(1)x.$$

$\square$

Exercise 4.2 (Radioactive Decay) Solve

$$f(x+y) = f(x)f(y),$$

where f is continuous/bounded.

5. Using functional equation to define elementary functions

One of the applications of functional equations is that they can be used to characterizing the elementary functions. In the following, you are provided exercises for the functional equations for the functions a^x, $\log_a x$, $\tan x$, $\sin x$, and $\cos x$. Can you setup the functional equations for $\cot x$, $\sec x$, $\csc x$, and hyper-trigonometric functions?

Exercise 5.1 The logarithmic function satisfies the property

$$\log(xy) = \log x + \log y$$

for any positive real numbers. Solve the equation

$$f(xy) = f(x) + f(y),$$

where both x, y are positive real numbers, where f is continuous/bounded.

Exercise 5.2 Solve the equation

$$f(x+y) = f(x)f(y)$$

where x, y are any real numbers, where f is continuous/bounded.

Exercise 5.3 Solve the equation

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$$

for any real numbers x, y, where f is continuous/bounded.

Exercise 5.4 Solve the equations for continuous/bounded functions $f(x), g(x)$

$$f(x+y) = f(x)g(y) + f(y)g(x)$$
$$g(x+y) = g(x)g(y) - f(x)f(y)$$

for any real numbers x, y.

6. Additional Problems

Exercise 6.1 Assume that $a_0 = 1$, $a_1 = 2$, and
$$a_n = 4a_{n-1} - a_{n-2}$$
for $n \geq 2$. Find a prime factor of a_{2015}.

This problem is, obviously, obsolete. By th year 2016 is also special. It is the dimensional of the space of all $2^6 \times 2^6$ skew-symmetric matrices. Just FYI.

Exercise 6.2 Solving
$$f(x) + f(\frac{x-1}{x}) = 1 + x, \quad x \neq 0, 1$$

Exercise 6.3 Exercise 2.3 revisited: is the solution unique?

Exercise 6.4 If we drop the assumption that f is continuous or boundedness in Example 4.1, is the function f still linear?

7. Solutions of the Exercises

Solution of Exercise 2.2: Replacing $x-1$ by x, we obtain

$$af(x)+bf(-x)=c(x+1).$$

In the above equation, if we replace x by $-x$, we get

$$af(-x)+bf(x)=c(-x+1).$$

Thus we solve

$$f(x)=\frac{ac(x+1)-bc(-x+1)}{a^2-b^2}.$$

After verification, the above is indeed the solution. □

Solution of Exercise 2.3: By replacing x with x^{-1}, we obtain

$$(1+f(x))(f(x^{-1})-(f(x^{-1}))^{-1})=\frac{(x-a)(1-ax)}{x}=(1+f(x^{-1}))(f(x)-(f(x))^{-1}).$$

Thus we have

$$f(x)=f(x^{-1}),$$

and hence f is determined by the equation

$$(1+f(x))(f(x)-(f(x))^{-1})=\frac{(x-a)(1-ax)}{x}.$$

□

Solution of Exercise 3.2:

$$a_n=2^n-1.$$

□

Solution of Exercise 4.2: We first conclude that

$$f(x_1+\cdots+x_n)=f(x_1)\cdots f(x_n).$$

By taking all x_i to be the same, we obtain

$$f(nx)=(f(x))^n.$$

In particular,

$$f(\frac{1}{m})=(f(1))^{\frac{1}{m}}.$$

And therefore we have

$$f(\frac{n}{m})=(f(1))^{\frac{n}{m}}.$$

The continuity/boundedness ensures that the above equation is also valid for irrational numbers.

□

Solution of Exercise 5.1: Let $g(x) = f(e^x)$. Then

$$g(x+y) = f(e^x \cdot e^y) = f(e^x) + f(e^y) = g(x) + g(y).$$

The problem is reduced to Example 4.1. □

Solution of Exercise 5.2: Either $f(x) \equiv 0$ or $f(x)$ is positive. So we can take $g(x) = \log f(x)$ in case $f(x)$ is positive. Then we reduce to Example 4.1. □

Solution of Exercise 5.3: Take $g(x) = \arctan f(x)$. □

Solution of Exercise 5.4: Take $h(x) = g(x) + \sqrt{-1} f(x)$. Then we obtain

$$h(x+y) = h(x)h(y).$$

Then using Exercise 4.2. □

8. Solutions of the additional problems

Solution of Exercise 7.1:

$$a_n = \frac{1}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n).$$

Since

$$(2+\sqrt{3})^{5n} + (2-\sqrt{3})^{5n} = ((2+\sqrt{3})^5 + (2-\sqrt{3})^5) \cdot \text{integer},$$

and since $(2+\sqrt{3})^5 + (2-\sqrt{3})^5 = 724 = 181$, a_{2015} is divisible by the prime number 181. □

Solution of Exercise 7.2: Replacing x by $\frac{x-1}{x}$, we obtain

$$f(\frac{x-1}{x}) + f(-\frac{1}{x-1}) = 1 + \frac{x-1}{x}.$$

Replacing x by $-\frac{1}{x-1}$, we obtain

$$f(-\frac{1}{x-1}) + f(x) = 1 - \frac{1}{x-1}.$$

Thus we have

$$f(x) = \frac{1}{2}(1 + x + 1 - \frac{1}{x-1} - 1 - \frac{x-1}{x}) = \frac{1}{2}(x + \frac{1}{x} - \frac{1}{x-1}).$$

□

Solution of Exercise 7.3: Yes, the solution is unique, even in general a cubic equation has three real roots.

This is because if x is small, the cubic equation only has one real root.

□

Solution of Exercise 7.4: The set of real numbers $\mathbb{R}$ can be considered as a vector space over rational numbers $\mathbb{Q}$. A basis for such a vector space is called a *Hamel basis*. That is, there is a set of real numbers u_α such that for any real numbers β, there is a unique representation of the form

$$\beta = \sum_{i=1}^{N} r_i u_{\alpha_i}.$$

The existence of such a basis is equivalent to the *Axiom of Choice*. Without assuming the continuity or boundedness, we can still prove that f is a *linear transformation* when regarding $\mathbb{R}$ as a vector space over $\mathbb{Q}$. However, one can prescribe any value on the Hamel basis to construct a (unbounded) solution. So in conclusion, the solution is not unique without the additional assumption. □

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