UCI Math Circle
Nov 14, 2016

1. We are going to play a game. In this game there are a number of coins, and two players take turns picking them up. Whoever gets the last coin wins. We will vary the rules a little so we actually have several slightly different games.

   Goals:

   - For each game, figure out whether it is better to go first or second
   - describe how you should play to make sure you win.

Game 1 On each turn, each player must take exactly one coin.

   i. Start with 2 coins.

   ii. Start with 3 coins.

   iii. Start with 4 coins.

   iv. After counting the number of coins, how do you decide whether you want to go first?

Game 2 On each turn, each player takes either one or two coins.

   i. Start with 2 coins.

   ii. Start with 3 coins.

   iii. Start with 4 coins.

   iv. Start with 5 coins.

   v. Start with 6 coins.

   vi. Start with 7 coins.

   vii. Start with 8 coins.

   viii. After counting the number of coins, how do you decide whether you want to go first?

   (Hint: Using your work above, you know some **winning positions** for player two, i.e. situations in which player two can win. You may want to think of how to leave the game in such a position after your turn.)

Game 3 On each turn, each player either takes one coin or changes one tails coin to a heads coin by flipping it over.

i. Start with 2 heads coins.

ii. Start with 1 heads and 1 tails.

iii. Start with 2 tails.

iv. Try all the different possible starting arrangements with 3 coins.

v. Now suppose you're allowed to decide who goes first after counting the number of coins, and seeing how many are heads. Can you find a pattern that gives a strategy to always win?

Game 4 There are 2 distinct piles of coins, and on each turn each player removes any number of coins from a single pile.

(a) Start with piles of 1 and 2 coins.

(b) Start with piles of 2 and 2 coins.

(c) Start with piles of 3 and 2 coins.

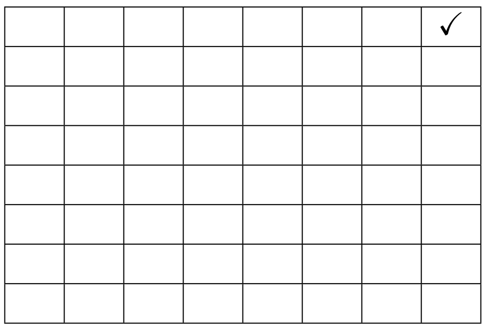
(d) Start with piles of 3 and 10 coins.

(e) Now suppose you're allowed to decide who goes first after counting the number of coins in each pile. Can you find a pattern that gives a strategy to always win? (This last game is a special case of a class of games called Nim. The strategy becomes more complicated when a third pile is introduced.)

2. More games with coins:

   (a) There is a coin placed at the bottom-left corner of an 8×8 chess board. On each turn, each player moves the coin any number of spaces to the right or up, but not both, and they must move at least one square. Whoever moves the coin into the top-right corner of the board wins. Can you find a strategy for the first or second player to win this game?

   (Hint: Having the coin in the top-right corner after your turn is clearly a winning position. Which other positions on the board are winning positions at the end of your turn? You may want to mark them on the board below.)



   (b) Two players take turns placing coins on a perfectly round table, without overlapping with any coin that has already been placed. The last player to place a coin wins. Can you find a strategy for the first or second player to win this game? (Hint: Think about the symmetry of the round table.)

3. Suppose there are 18 coins on a table: 7 heads and 11 tails. You are blindfolded and can't tell in any way whether a particular coin is heads or tails, but you can flip coins over if you wish.

   How can you divide the coins into two groups such that each group contains the same number of heads (but not necessarily the same number of tails)?

4. A piece of string is cut in two at a point selected at random. What is the probability that the longer pieces is at least x times as long as the shorter piece ($x \geq 1$)?

5. In the Clock Game on the game show *The Price is Right*, a contestant must guess the price (rounded to the nearest dollar) of a prize which is worth less that $2,000. After each guess the contestant is told whether the guess was correct, too high, or two low. Assume the contestant is mathematically savvy but has no idea how much the prize is worth. With how many guesses is she guaranteed to win the prize?

6. (a) Jane has two children, and they are not twins. At least one of her children is a boy. What are the chances that her other child is also a boy?

   (b) Greg has two children, and they are not twins. His youngest child is a girl. What are the chances that the other child is a girl?

   (The answers to these two questions are not the same!)

7. On the game show *Let's Make a Deal*, there is a game in which a contestant chooses from among three doors. Behind one door is a car and behind the other two doors are goats. The host of the show knows what is behind each door and after the contestant chooses one, the host will always open one of the two remaining doors revealing a goat. Then he will offer the contestant to stick with their original choice or switch to the other remaining door. What are the contestant's chances of winning the car if they stick with their original choice? If they switch? What should they do?