Sum of Squares and Difference Calculus

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1 Introduction:

One of mathematics most famous stories is of the legendary mathematician Gauss as an elementary school student. One day, Gauss's teacher wanted to take a nap instead of teach his class. To keep his students busy while he slept, he asked them to sum the first 100 numbers. After only a few seconds of work, young Gauss surprised his teacher by approaching him with the correct answer.

Gauss had written $S = 1 + 2 + \cdots + 100$ on one line and $S = 100 + 99 + \cdots + 1$ directly underneath. Adding the two lines, Gauss got

$$\begin{array}{rrcrcrcccr} & \text{S} & = & 1 & + & 2 & + & \cdots & + & 100 \\ + & \text{S} & = & 100 & + & 99 & + & \cdots & + & 1 \\ \hline & \text{2S} & = & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

So $2S = 101 \cdot 100$. Thus, $S = 5050$.

Tonight, using Gauss's clever semi-visual proof as motivation, we will use triangles to prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Afterwards, we will develop techniques from difference calculus that will allow us to quickly compute both $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$.

2 An Introduction to Sigma Notation

Tonight you will need to use Σ-notation to evaluate sums. This first section introduces that notation. If you feel comfortable with Σ-notation, then feel free to skip to the next section.

Let $a_1, a_2, ..., a_n$ be an ordered list of numbers. The subscript on the a tells us where that number appears in the list. For example, a_5 is the fifth number in the list because the subscript on the a is 5.

Usually you will also be given a formula for the k^{th} number in the list. For example we can let $a_k = k$ for any number k between 1 and n (so the k^{th} number in the list is equal to k.) In that case, if $k = 1$, then $a_1 = 1$, if $k = 2$, then $a_2 = 2$, if $k = 3$, then $a_3 = 3$, and so on. So the list $a_1, a_2, ...a_n$ becomes $1, 2, ...n$. Let's do another example.

Example 2.1 *Let $a_k = k^2$. What are the numbers a_1, a_2, a_3, a_4, a_5.*

From the formula, $a_1 = 1^2 = 1$, $a_2 = 2^2 = 4$, $a_3 = 9$, $a_4 = 16$ and $a_5 = 25$.

Problem 2.2 *Let $a_k = k/2$, $b_k = 1$, $c_k = a_k + b_k$. What are the numbers a_1, a_2, a_3, a_4? How about $b_1, b_2, b_3,$ b_4 and c_1, c_2, c_3, c_4?*

Σ is the Greek letter Sigma and Σ-notation provides us with a concise way to write sums. Given an ordered list $a_1, a_2, a_3, ...a_n$, $\sum_{k=m}^{n} a_k$ tells us to add the numbers in the list starting with a_m and ending with a_n. Let's do several examples.

Example 2.3 *Consider the list a_1, a_2, a_3, a_4, a_5 where $a_k = k$. What is $\sum_{k=1}^{5} a_k$?*

$\sum_{k=1}^{5} a_k$ tells us to add the first through fifth numbers in our list. So

$$\begin{aligned} \sum_{k=1}^{5} a_k &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 1 + 2 + 3 + 4 + 5 \\ &= 15. \end{aligned}$$

Problem 2.4 *Consider the list* $a_1, a_2, a_3,\ a_4$*. What is* $\sum_{k=1}^{4} a_k$ *when* $a_k = k^2$*? How about when* $a_k = 1$*?* $a_k = k + 2$*?*

To make our notation shorter, if we have a formula for a_k we will often replace the a_k next to the $\boldsymbol{\Sigma}$ with the formula.

For example, instead of saying $\sum_{k=1}^{5} a_k$ when $a_k = k$, we just say $\sum_{k=1}^{5} k$.

Example 2.5 *Compute* $\sum_{k=1}^{4} k^2$.

In this example, our list is a_1, a_2, a_3, a_4 with $a_k = k^2$. So,

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

because $a_1 = 1^2$, $a_2 = 2^2$, $a_3 = 3^2$, $a^4 = 4^2$ and $a_k = k^2$.

Problem 2.6 *Compute* $\sum_{k=1}^{3} k^2$*,* $\sum_{k=1}^{4} 1$*,* $\sum_{k=1}^{7} k$*, and* $\sum_{k=2}^{4} 2k + 1$*.*

Problem 2.7 *Gauss showed that for any number $n > 1$ that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. What is $\sum_{k=1}^{10} k$? $\sum_{k=1}^{50} k$? $\sum_{k=1}^{100} k$?*

3 A Triangular Argument that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Let's start by considering the Table 1 shown below:

Table 1

k	0	1	2	3	4	5	6	7
k^2	0	1	4		16			49
$(k+1)^2 - k^2$	1	3	5				13	

Problem 3.1 *Fill in the blank entries in Table 1 with the correct values.*

Now, notice that

$$\begin{aligned} 2^2 &= 4 = 1 + 3 \\ 3^2 &= 9 = 1 + 3 + 5 \\ 4^2 &= 16 = 1 + 3 + 5 + 7. \end{aligned}$$

Do you see where those numbers appear in the above table? Every number in the second row is the sum of the numbers in the third row that are to the left of it. That's interesting. Let's present the information in the table in a different way.

Problem 3.2 *Fill in the missing values in the triangle below. For each row, how many numbers are there to the right of the "=" sign? Let k be any whole number. For the k^{th} row, can you write down a formula for the last blank in the row in terms of the variable k? (This will help you fill in the last blank.)*

$$\begin{aligned}
&1^2 = 1 \\
&2^2 = 1 + 3 \\
&3^2 = 1 + 3 + 5 \\
&4^2 = _ + _ + 5 + _ \\
&5^2 = _ + _ + _ + _ + _ \\
&\vdots \qquad\qquad\qquad\qquad \ddots \\
&n^2 = 1 + 3 + 5 + 7 + 9 + 11 + \cdots + ___
\end{aligned}$$

Problem 3.3 *From the last row of the triangle, we have that* $n^2 = 1 + 3 + 5 + \cdots + (2n-1)$*. Rewrite that equation in* Σ*-notation.*

Great! We showed visually that $n^2 = \sum_{k=1}^{n}(2k-1)$. Let's see if we can show that formula another way. Take another look at the last row of Table 1. The last row consists of consecutive odd numbers.

Problem 3.4 *From Table 1, why does it appear that* $(k+1)^2 - k^2 = 2k+1$*?*

Problem 3.5 *Verify using algebra that* $(k+1)^2 - k^2 = 2k+1$*. Also show that* $k^2 - (k-1)^2 = 2k-1$*.*

So we have that $\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} k^2 - (k-1)^2$. Let's see if we can show algebraically that this equals n^2. This is a little tricky so let's do an example first to show how we might prove it.

Example 3.6 *Suppose* $n = 5$*. Show that* $\sum_{k=1}^{5} k^2 - (k-1)^2 = 5^2$*.*

Notice that

$$\begin{aligned}
\sum_{k=1}^{5} k^2 - (k-1)^2 &= (1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + (4^2 - 3^2) + (5^2 - 4^2) \\
&= \cancel{1}^2 - 0^2 + \cancel{2^2} - \cancel{1^2} + \cancel{3^2} - \cancel{2^2} + \cancel{4^2} - \cancel{3^2} + 5^2 - \cancel{4^2} \\
&= 5^2 - 0^2 \\
&= 5^2.
\end{aligned}$$

See how most of the terms in the middle canceled? That was a nice simplification. Now let's see if we can generalize this for any number n.

Problem 3.7 *Using the previous example as a guide, compute* $\sum_{k=1}^{7} k^2 - (k-1)^2$. *Show that* $\sum_{k=1}^{n} k^2 - (k-1)^2 = n^2$. *Conclude that* $\sum_{k=1}^{n} (2k-1) = n^2$.

So far we have shown that $\boxed{n^2 = \sum_{k=1}^{n} 2k - 1.}$ This formula will be very useful later. Now let's turn to something else.

Consider Triangle 1, 2 and 3 shown in the other packet. The triangles have n numbers along the base and n numbers along the height.

Problem 3.8 *How many numbers are there in these triangles? Is it the same as the area of an* $n \times n$ *right triangle? If not, why are they different?*

Problem 3.9 *What is the sum of all of the numbers in Triangle 1? Write your answer in* Σ*-notation. How about Triangle 2 and Triangle 3?*

Let's define what it means to add two triangles. If we have two triangles T_1 and T_2 that are the same size, then we say that $T_1 + T_2$ is the triangle where each entry is the sum of the corresponding entries of T_1 and T_2. For example,

$$\begin{array}{lll} 1 & & \\ 1 \quad 1 & & \\ 1 \quad 1 \quad 1 & & \end{array} \quad + \quad \begin{array}{lll} 2 & & \\ 2 \quad 2 & & \\ 2 \quad 2 \quad 2 & & \end{array} \quad = \quad \begin{array}{lll} 3 & & \\ 3 \quad 3 & & \\ 3 \quad 3 \quad 3 & & \end{array}$$

and

$$\begin{array}{lll} 1 & & \\ 5 \quad 3 & & \\ 2 \quad 0 \quad 1 & & \end{array} \quad + \quad \begin{array}{lll} 2 & & \\ 4 \quad 2 & & \\ 6 \quad 7 \quad 0 & & \end{array} \quad = \quad \begin{array}{lll} 3 & & \\ 9 \quad 5 & & \\ 8 \quad 7 \quad 1 & & \end{array}$$

Problem 3.10 *What is the sum of the following triangles?*

$$\begin{array}{lll} 3 & & \\ 4 & 5 & \\ 6 & 7 & 1 \end{array} \quad + \quad \begin{array}{lll} 0 & & \\ 0 & 1 & \\ 2 & 2 & 6 \end{array}$$

Problem 3.11 *Recall that Triangle 1, Triangle 2 and Triangle 3 are the three triangles shown in the other packet. What is Triangle 1 + Triangle 2 + Triangle 3? Draw the triangle.*

Let Triangle 4 be the $n \times n$ right triangle with $2n + 1$ in each entry as shown below:

$$
\begin{array}{llllllll}
2n+1 & & & & & & & \\
2n+1 & 2n+1 & & & & & & \\
2n+1 & 2n+1 & 2n+1 & & & & & \\
2n+1 & 2n+1 & 2n+1 & 2n+1 & & & & \\
2n+1 & 2n+1 & 2n+1 & 2n+1 & 2n+1 & & & \\
\vdots & & & & & \ddots & & \\
2n+1 & \cdots & & & & \cdots & 2n+1 & \\
2n+1 & 2n+1 & & & & \cdots & & 2n+1
\end{array}
$$

Triangle 4

Problem 3.12 *What is the sum of all of the numbers in Triangle 4?*

In problem 3.11, you should have found that sum of Triangle 1, 2, and 3 is Triangle 4, and in problem 3.9 you found that the sum of the numbers in Triangle 1 is $\sum_{k=1}^{n} k^2$. Since Triangle 2 and Triangle 3 are rearrangements of Triangle 1, the sum of the numbers in each of those triangles is also $\sum_{k=1}^{n} k^2$.

Problem 3.13 *Conclude that* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

That proof was pretty clever and coming up with clever proofs is rewarding. However, this proof doesn't provide us with a general way to easily compute $\sum_{k=1}^{n} k^m$ where $m > 2$. In the next section, we will develop techniques of difference calculus that will make it much easier to compute $\sum_{k=1}^{n} k^m$ where m is $1, 2, 3$ or bigger.

4 Difference Calculus and the Sum of Squares and Sum of Cubes

Definition 4.1 *For any number x and any non-negative integer m, define $x^{\underline{m}}$ by*

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1).$$

In English, we say that $x^{\underline{m}}$ is "x to the m falling."

For example,

$$5^{\underline{2}} = 5(5-1) = 20.$$

Problem 4.2 *Show that $5^{\underline{3}} = 60$ and $5^{\underline{4}} = 120$. Also calculate $(-3)^{\underline{2}}, (-3)^{\underline{3}}$ and $(-3)^{\underline{4}}$.*

In the previous problem you may have noticed that $5^{\underline{2}} < 5^{\underline{3}} < 5^{\underline{4}}$. It seems as though if $m < n$, then $5^{\underline{m}} < 5^{\underline{n}}$. Is that true?

Problem 4.3 *Show that this is not always true. That is, give an example of two numbers m and n so that $m < n$ but $5^{\underline{m}} > 5^{\underline{n}}$.*

You also may have noticed in problem 4.2 that $(-3)^{\underline{2}}$ and $(-3)^{\underline{4}}$ are positve while $(-3)^{\underline{3}}$ is negative. So I'm going to conjecture that if $k^{\underline{m}}$ is negative, then $k < 0$ and m is odd. Is that conjecture true?

Problem 4.4 *Show that the conjecture "if $k^{\underline{m}}$ is negative, then $k < 0$ and m is odd" is false. Show that if we also require that k be an integer, then the conjecture is true.*

Nicely done. Here are a few more problems to get used to the falling powers.

Problem 4.5 *What is $x^{\underline{1}}$?*

Problem 4.6 *Write k^2 and k^3 as sums of falling powers.*

We will now introduce the definition of the difference operator Δ which will be essential to computing $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$.

Definition 4.7 *Define the difference operator Δ by*

$$\Delta f(x) = f(x+1) - f(x).$$

Let's do a couple of examples to get used to this definition.

Example 4.8 *Now let's suppose that $g(x) = x$. What is $\Delta g(x)$.*

$$\Delta g(x) = \Delta x = (x+1) - x = 1.$$

Example 4.9 *Let's suppose that $f(x) = x^2$. What is $\Delta f(x)$? Well,*

$$\Delta f(x) = (x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1,$$

so $\Delta f(x) = 2x + 1$.

Problem 4.10 *Suppose that $h(x) = x^{\underline{2}}$. Then $\Delta h(x) = (x+1)^{\underline{2}} - x^{\underline{2}}$. Show that $\Delta h(x) = 2x$. Rewrite $2x$ in terms of falling powers.*

Problem 4.11 *Let $j(x) = x^{\underline{3}}$. Show that $\Delta j(x) = 3x(x-1)$. Rewrite $3x(x-1)$ in terms of falling powers.*

In the previous two problems you showed that

$$\Delta x^{\underline{2}} = 2x^{\underline{1}}$$

$$\Delta x^{\underline{3}} = 3x^{\underline{2}}.$$

If you do a few more calculations then you can show that

$$\Delta x^{\underline{4}} = 4x^{\underline{3}}$$

and

$$\Delta x^{\underline{5}} = 5x^{\underline{4}}.$$

There seems to be a pattern here.

Problem 4.12 *Show that for any number m that $\Delta x^{\underline{m}} = mx^{\underline{m-1}}$.*

At this point, it is not clear how the difference operator and the falling powers relate to $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$. So let's do a few examples to illustrate the relationship between Δ and Σ.

Example 4.13 *Let's calculate* $\sum_{x=2}^{4} \Delta f(x)$.

Since $\Delta f(x) = f(x+1) - f(x)$,

$$\begin{aligned}
\sum_{x=2}^{4} \Delta f(x) &= \sum_{x=2}^{4} f(x+1) - f(x) \\
&= f(2+1) - f(2) + f(3+1) - f(3) + f(4+1) - f(4) \\
&= f(3) - f(2) + f(4) - f(3) + f(5) - f(4) \\
&= \cancel{f(3)} - f(2) + \cancel{f(4)} - \cancel{f(3)} + f(5) - \cancel{f(4)} \\
&= f(5) - f(2).
\end{aligned}$$

That simplified nicely. Now it is your turn to try something similar.

Problem 4.14 *Using the previous example as a guide, show that* $\sum_{x=1}^{5} \Delta f(x) = f(6) - f(1)$. *Let* a *be any number. Show that* $\sum_{x=a}^{a+4} \Delta f(x) = f(a+5) - f(a)$.

Notice that in both the previous example and problem, after you expanded the sum, all of the middle terms canceled each other out. That's important because it can help us get a general formula relating $\sum$ and Δ.

Problem 4.15 *Let a and b be two numbers so that $a < b$. Show that*

$$\boxed{\sum_{x=a}^{b} \Delta f(x) = f(b+1) - f(a).}$$

It's almost as if the boxed formula says that the Σ-operator undoes the Δ-operator. Cool. Let's practice using the boxed formula.

Problem 4.16 *Use the boxed formula in the previous problem to find a formula for $\sum_{k=a}^{b} 2^k$ (Do not use the geometric series formula).*

Now let's connect the discussion of Δ and Σ to $x^{\underline{m}}$. In Problem 4.12, you showed that $\Delta x^{\underline{n+1}} = (n+1)x^{\underline{n}}$. Thus,

$$\frac{\Delta x^{\underline{n+1}}}{n+1} = x^{\underline{n}}.$$

So,

$$\sum_{x=a}^{b} x^{\underline{n}} = \sum_{x=a}^{b} \frac{\Delta x^{\underline{n+1}}}{n+1} = \frac{1}{n+1} \sum_{x=a}^{b} \Delta x^{\underline{n+1}}.$$

Problem 4.17 *Using the previously boxed formula what does this sum equal?*

Good job. In summary, the two important formulas that you need to remember are

$$\boxed{\sum_{x=a}^{b} \Delta f(x) = f(b+1) - f(a)} \text{ and } \boxed{\sum_{x=a}^{b} x^{\underline{n}} = \frac{1}{n+1}(b+1)^{\underline{n+1}} - \frac{1}{n+1} a^{\underline{n+1}}.}$$

Now that you know the two formulas written above, you are now ready to give formulas for $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$.

Problem 4.18 *Using the two boxed formulas show that*

$$\sum_{k=1}^{100} k = 5050 \textit{ and } \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Problem 4.19 *Give formulas in terms of n for $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$.*

4.1 Challenge Problem

Problem 4.20 *Find the number of ways to represent* 1050 *as a sum of consecutive positive integers. (*1050 *by itself counts as one way.)*

5 Connections to Calculus

For those who are currently taking or have taken differential/integral (regular) calculus, many connections can be made between that and difference calculus. The difference operator Δ and the derivative $\frac{d}{dx}$ are analogous and have similar power rules.

$$\Delta x^{\underline{m}} = mx^{\underline{m-1}} \text{ and } \frac{d}{dx}x^m = mx^{m-1}.$$

In differential/integral calculus the derivative and the integral are inverse operations that satisfy two fundamental theorems of calculus, one fundamental theorem for indefinite integrals and one for definite integrals. Tonight we derived a the fundamental theorem of difference calculus which is analogous to calculus's fundamental theorem for definite integrals; it is the boxed equation that relates Δ and Σ. For more information on difference calculus, see the references.

References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, 1989.

[2] David F. Gleich, "Finite Calculus: A tutorial for solving nasty sums". Accessed on September 22, 2016. Retrieved from https://www.cs.purdue.edu/homes/dgleich/publications/Gleich

[3] "Square pyramidal number", Wikipedia: The Free Encyclopedia, https://en.wikipedia.org/wiki/Square$_pyramidal_number$

[4] NRICH Team, "Clever Carl". Accessed on September 22, 2016. Retrieved from https://nrich.maths.org/2478.

Triangle 1

$$
\begin{array}{|llllllll}
1 & & & & & & & \\
1 & 3 & & & & & & \\
1 & 3 & 5 & & & & & \\
1 & 3 & 5 & 7 & & & & \\
1 & 3 & 5 & 7 & 9 & & & \\
\vdots & & & & & \ddots & & \\
1 & \cdots & & & & \cdots & 2n-3 & \\
1 & \cdots & & & & \cdots & 2n-3 & 2n-1 \\
\hline
\end{array}
$$

Triangle 2

$$
\begin{array}{|llllllll}
2n-1 & & & & & & & \\
2n-3 & 2n-3 & & & & & & \\
\vdots & & \ddots & & & & & \\
9 & \cdots & \cdots & 9 & & & & \\
7 & \cdots & \cdots & 7 & 7 & & & \\
5 & \cdots & \cdots & 5 & 5 & 5 & & \\
3 & \cdots & \cdots & 3 & 3 & 3 & 3 & \\
1 & \cdots & \cdots & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$

$$
\text{Triangle 3}\quad
\left|\begin{array}{cccccccc}
1 & & & & & & & \\
3 & 1 & & & & & & \\
5 & 3 & 1 & & & & & \\
7 & 5 & 3 & 1 & & & & \\
9 & 7 & 5 & 3 & 1 & & & \\
\vdots & & & & & \ddots & & \\
2n-3 & \cdots & & & & \cdots & 1 & \\
2n-1 & 2n-3 & & & & \cdots & & 1 \\
\hline
\end{array}\right.
$$