UCI MATH CIRCLE: POPULATION DYNAMICS

ADAPTED FROM A SESSION OF THE UNIVERSITY OF UTAH MATH CIRCLE

When Europeans first arrived in the Americas, Passenger Pigeons were perhaps the most populous bird in the area. At that time, the size of the Passenger Pigeon population was somewhere between 3 to 5 billion. Residents of the Eastern United States wrote that There are wild pigeons in winter beyond number or imagination. . . I myself have seen three or four hours together flocks in the air, so thick that even have they shadowed the sky from us. Despite the enormous number of birds present in the United States in the 1700s, there are no Passenger Pigeons left on earth today, as the last known bird (Martha) died in captivity in 1914. The Passenger Pigeon was hunted and killed by humans for many years before its extinction. During this time, the number of pigeons in the United States decreased, but no one seemed to expect the birds to disappear completely. Could scientists have predicted the extinction of the Passenger Pigeon using mathematics? Can we accurately model how population numbers will change if we change the external factors? Well start to answer these questions by looking at some mathematical models.

1. Pond 1

First, consider a pond that is inhabited by one type of fish. What factors determine how many fish are in the pond from one day (month, year) to the next? In real life, the number of fish in the pond at some later time probably depends on a variety of factors including the number of fish currently in the pond, the amount of food available to the fish, the number of predators that eat the fish, and the environmental conditions of the area around the pond (temperature, pollution levels). Instead of thinking of the exact number of fish in the pond, we can also consider how full the pond is. We introduce a variable N denoting the fraction of the pond that contains fish: for example, if $N = 0$ the pond is empty, if $N = 1/2$ the pond is half full, if $N = 1$ the pond is completely full of fish. Because we want to track the fullness of the pond over time, we write N_k to represent the fraction of the pond that is full of fish k months from now.

Assume that the fraction of the pond that is full of fish one month from now (represented by N_1) depends on how full the pond currently is (N_0) and the following equation:

$$N_1 = \frac{2N_0}{1 + 2N_0}$$

Remember that we are thinking of only how full the pond is, or alternatively of the fraction of the pond that contains fish. Therefore, N_0 and N_1 must be between 0 and 1.

In general, say we had modeled the fullness of the pond at time $t+1$ by

$$N_{t+1} = \frac{2N_t}{1+2N_t}$$

Exercise 1.1. Find N_1, N_2, N_3 if:

$$N_0 = 0 \qquad N_1 = \qquad N_2 = \qquad N_3 =$$

$$N_0 = 1/4 \qquad N_1 = \qquad N_2 = \qquad N_3 =$$

$$N_0 = 1/2 \qquad N_1 = \qquad N_2 = \qquad N_3 =$$

$$N_0 = 3/4 \qquad N_1 = \qquad N_2 = \qquad N_3 =$$

$$N_0 = 1 \qquad N_1 = \qquad N_2 = \qquad N_3 =$$

Exercise 1.2. Using the data points from Exercise 1.1, plot the function

$$f(x) = \frac{2x}{1+2x}$$

between 0 and 1.

Writing it this way, we see that

$$N_1 = f(N_0)$$
$$N_2 = f(N_1) = f(f(N_0))$$
$$\ldots$$
$$N_t = f(f(f(\ldots f(N_0)))\ldots)$$

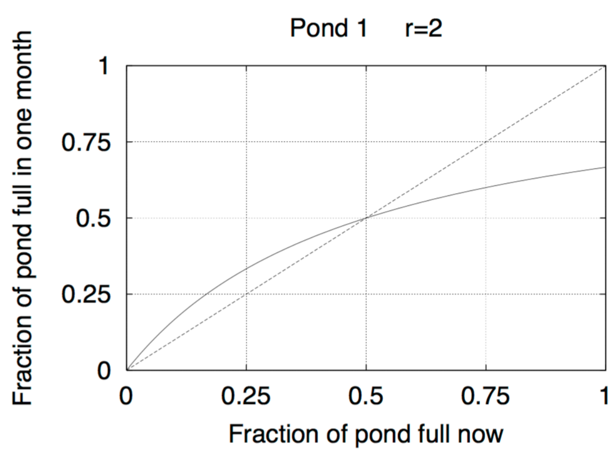
By the way, a setting where you apply the same function over and over again to the result is called a *dynamical system*.

Exercise 1.3. How full will the pond be after ten months if it starts out one quarter full?

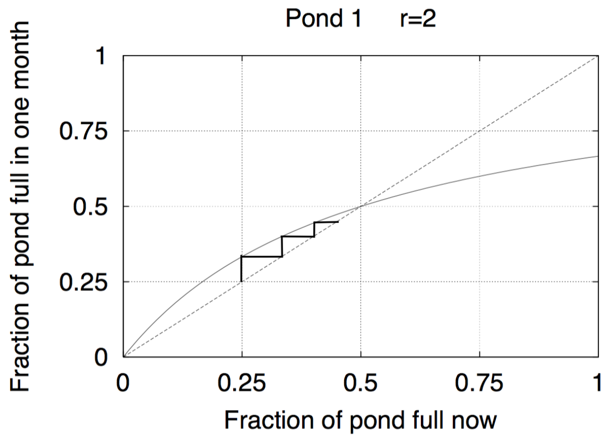
What about 50 years from now? This is becoming too much work. At this point, we have two options:

- Use a computer to run a simulation. (e.g. in Excel, this is a fun thing you can try at home)
- Do something smart. (preferred method)

1.1. **Cobwebbing.**



The first smart method is a visual way. Here, we have graphed our function $f(x)$ as well as the line $y = x$. Starting at $x = 0.25$, we see that $f(x) = 1/3$ by intersecting the curve with the vertical line $x = 0.25$, then find the line $x = f(0.25)$ by intersecting the line $y = x$ with the horizontal line at the point $(0.25, f(0.25))$. We then find $f(f(x))$ by $\ldots$ and keep repeating. This is called cobwebbing.



Exercise 1.4. How full will the pond be after 100 years if we start with $N_0 = 0.25$? What about $N_0 = 1$, $N_0 = 1/3$.

Exercise 1.5. Some values x, are *equilibrium points* of this dynamical system in the sense that if the fullness of the pond is at those x values, then the fullness stays the same forever. How many equilibrium points does this dynamical system have? What values of x are they?

1.2. **Using algebra.** We can also find the equilibrium points using algebra.

Exercise 1.6. Find the equilibrium points of the above dynamical system by solving the equation

$$N = \frac{2N}{1 + 2N}$$

1.3. **General r.** Now let's look at a system where we have a variable that controls how hard it is for the fish to survive. We call this variable r (maybe for "rough life index"?). Let us set:

$$N_{t+1} = \frac{rN}{1+2N}$$

Exercise 1.7. Using the techniques you learned above, predict the fate of the fish after 10 years (120 months) for the following starting scenarios.

$\mathrm{r} = 1/2 \quad \mathrm{N}_0 = 0, \qquad \mathrm{N}_{120} \approx$

$\mathrm{N}_0 = 1/4, \quad \mathrm{N}_{120} \approx$

$\mathrm{N}_0 = 1/2, \quad \mathrm{N}_{120} \approx$

$\mathrm{N}_0 = 3/4, \quad \mathrm{N}_{120} \approx$

$\mathrm{N}_0 = 1, \qquad \mathrm{N}_{120} \approx$

$r = 3/2 \quad N_0 = 0, \qquad N_{120} \approx$

$N_0 = 1/4, \quad N_{120} \approx$

$N_0 = 1/2, \quad N_{120} \approx$

$N_0 = 3/4, \quad N_{120} \approx$

$N_0 = 1, \qquad N_{120} \approx$

$r = 3 \quad N_0 = 0, \qquad N_{120} \approx$

$N_0 = 1/4, \quad N_{120} \approx$

$N_0 = 1/2, \quad N_{120} \approx$

$N_0 = 3/4, \quad N_{120} \approx$

$N_0 = 1, \qquad N_{120} \approx$

Exercise 1.8. Using algebra, figure out a rule for determining how full the pond will be after a long time for any value of r. When is there only one equilibrium point, and when are there multiple equilibrium points? (what is the equation you will need to solve?)

Think about what changing the value of r means biologically. How do the different values of r affect how full the pond is?

2. Pond 2

We will now analyze a slightly different model:

$$N_{t+1} = \frac{rN_t^2}{1 + 4N_t^2}$$

Exercise 2.1. Use cobwebbing to analyze what will happen after many years for $r = 3$, $r = 4$, and $r = 5$, and $N_0 = 1/8$ and $N_0 = 3/4$

Exercise 2.2. For $r = 3$, $r = 4$, $r = 5$ use algebra to find the equilibrium points. It is possible to have more than 2 equilibrium points.

Exercise 2.3. Using algebra, figure out a rule for determining how full the pond will be after a long time for any value of r. How many equilibrium points are there? Solve the equation

$$N = \frac{rN^2}{1 + 4N^2}$$

Challenge problems:

Exercise 2.4. Plot a graph of the equilibrium points for all values of r. This type of graph is called a bifurcation diagram.

Exercise 2.5. How does this relate to the story of the Passenger Pigeon that we talked about initially? What if r is inversely related to the hunting rate (i.e. large r implies low levels of hunting, and small r implies high levels of hunting)? If the model for Pond 2 also describes the Passenger Pigeon population, does the equilibrium population size drop gradually as hunting increases (r decreases), or is there a sudden drop when hunting increases past a critical point?