INTRODUCTION TO SIMPLICIAL COMPLEXES

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0.1. **Introduction.** In this activity set we are going to introduce a notion from *Algebraic Topology* called 'simplicial homology'. The main goal of this activity is to learn how to construct certain topological invariants of different objects using simplicial complexes. These are visual, mathematical structures that represent shapes we know well but that we are able to perform computations on. To outline, we will be

(1) Discussing **simplices**, these are the "building blocks" to create our complexes. We'll be exploring how to define, construct, and write down these structures in different ways.
(2) We then extend to building and **simplicial complexes**. This will take some tinkering and definitions, and activities to get comfortable.
(3) Then we begin relating these complexes to structures we know. We will look at objects and see how they can be approximated from a simplical complex. We show how we can use the **gluing operation** to create objects from our simplicial complexes.
(4) We introduce the **boundary operator**, ∂, which you can use to mathematically compute the boundary of a given surface or object (you can show, for example, that the boundary of a ball is most definitely a sphere!).

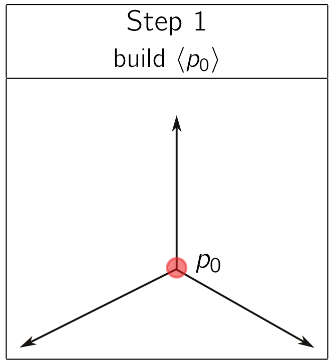
1. A SIMPLEX

An n-simplex is a geometric object with $(n+1)$ vertices which lives in an n-dimensional space (and cannot fit in any space of smaller dimension). The vertices of the simplex "generate" the simplex through a simple geometric construction which we illustrate below. The idea is easy: one vertex generates a point, two vertices generate a segment (by connecting the two points), three vertices generate a triangle (by connecting every pair of points with segments and filling the space in between) and so on. Notice how $(n+1)$ vertices are needed to generate an object of dimension n. Also note that because we want an n-simplex to be an object of dimension n, a bit of care must be exercised in the choice of its $(n+1)$ generating vertices. For example, three points which belong to the same line have no hope to generate a 2-dimensional object!
We now outline the steps for building an n-simplex (for $n = 0, 1, 2, 3$). These are the only simplexes we can visualize. The construction generalizes to simplices of bigger dimension, but we will need to rely on our imagination to picture the actual geometric objects. Using precise mathematical notation facilitates the abstraction and helps you think about higher dimensional structures or concepts that often can't even be visualized.

Definition 1.1 (n-simplexes, for $n = 0, 1, 2, 3$)**.**

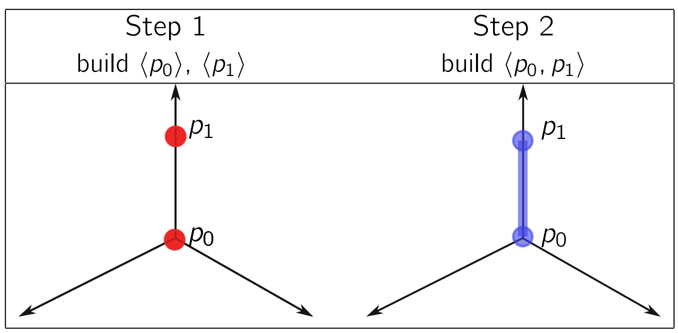
Start out with the 3-D space and draw three coordinate axes. Your axes do not necessarily have to be perpendicular to each other, just make sure they do not crush into a plane.
We will be denoting a general simplex by 'σ' and, in particular, an n-simplex by σ_n.

- **0-simplex** (a simplex $\langle p_0\rangle$ generated by one point, p_0)
  A 0-simplex is a point; for example, the origin or another point in the coordinate axis $\sigma_0 = \langle p_0\rangle$.

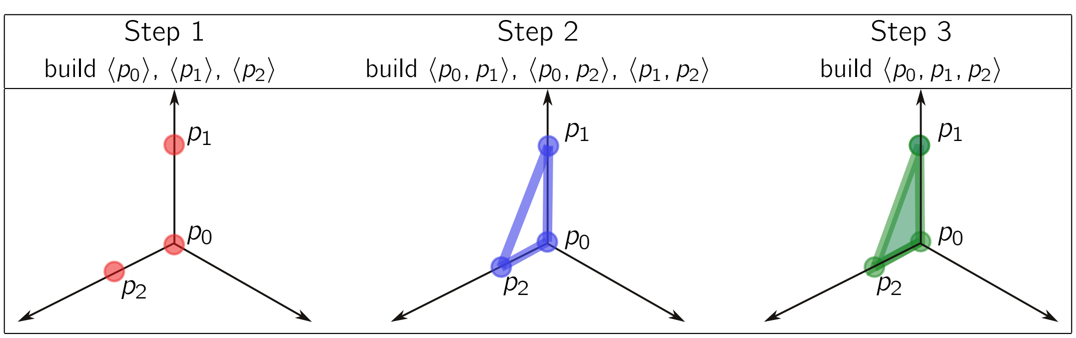


For simplicity, we will always take p_0 to be at the origin of our coordinate axes.

- **1-simplex** (a simplex $\langle p_0, p_1 \rangle$ generated by two points, p_0 and p_1)
  A 1-symplex is a line segment (including its end-points). To build one, take the origin and 1 *other* point which lies on a coordinate axis. This construction, produces two 0-subsimplices. Next, connect the two points to get your 1-simplex $\sigma_1 = \langle p_0, p_1 \rangle$.



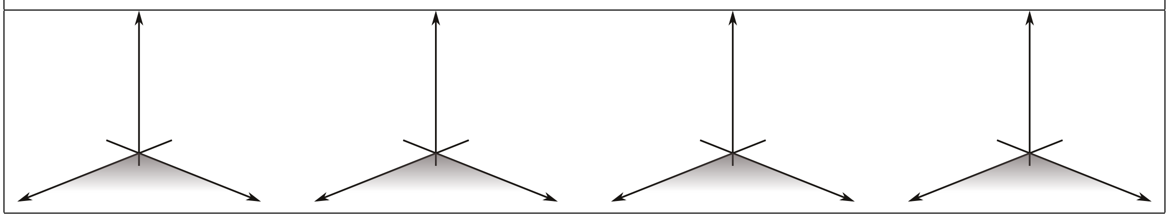
- **2-simplex** (a simplex $\langle p_0, p_1, p_2 \rangle$ generated by three points, p_0, p_1, p_2)
  A 2-simplex is a solid triangle (including its border). To build one, take the origin and 2 *other* points which lie on two different coordinate axis. So far, this gives you three 0-subsimplices. Next, connect all possible pairs of two points, to get three 1-subsimplices. Finally, fill in the resulting triangle to obtain your 2-simplex $\sigma_2 = \langle p_0, p_1, p_2 \rangle$.



- **3-simplex** (a simplex $\langle p_0, p_1, p_2, p_4 \rangle$ generated by three points, p_0, p_1, p_2, p_4)

Task 1. *Fill in the blanks and complete the pictures below.:*
A 3-simplex is a solid tetrahedron (including its border). To build one, take the origin and ___ *other* points which lie on ____ different coordinate axis. This construction produces ______________________. Next, connect all possible pairs of two points, to get ___ 1-subsimplices. The next step is to ___________ ____________________________, to obtain ______________________. Finally, fill in the resulting ______________________ to obtain your ___ -simplex, σ_3.

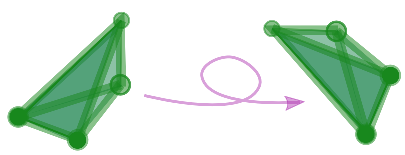
Step 1	Step 2	Step 3	Step 4



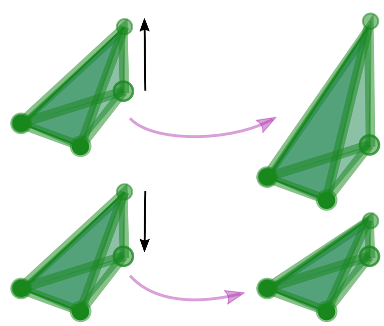
Definition 1.2 (n-simplex)**.** To construct an n-simplex for $n > 3$, iterate this construction.

A simplex, mathematically, doesn't have any fixed shape or size, or orientation. In particular, the following

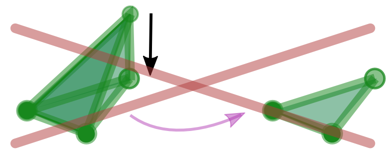
(1) **Rigid motions. (Rotate, translate, dilate)** We can move or rotate the simplex to anywhere we desire, and it still counts as the same simplex.



(2) **Stretch.** We can stretch out generating points away from each other (and change the connected structures too).



However, you cannot **crush** simplices- you cannot turn an n-simplex into an $(n-1)$-simplex by deforming it.

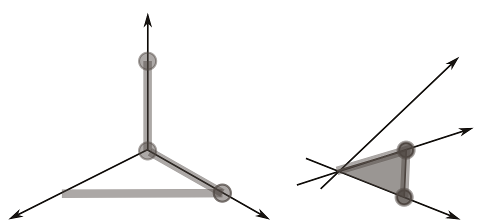


1.0.1. *A remarks about simplices.* The order in which we list the vertices generating a simplex does not matter, for example:

$$\langle a, b, c\rangle = \langle c, a, b\rangle = \langle b, c, a\rangle .$$

This fact holds true for any n-simplex (later, we'll be adding orientation to these structures, then the order *will* matter!).

Task 2. *What do the following structures need to become simplexes? What will be the dimension of each resulting simplex? We have included axes for your convenience!. First label the points and write out the components using the notation we have shown above to help see what you are missing.*



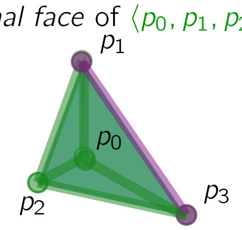
Definition 1.3 (face)**.** Let $\sigma = \langle p_0, p_1, \ldots, p_n\rangle$ be an n-dimensional simplex. A face of σ is a sub-simplex of σ, namely, the simplex generated by a subset of the vertices of σ. To get a face of σ of dimension $m \leq n$, choose $m+1$ points among $p_0, p_1, \ldots, p_n$ and take the corresponding simplex.

Task 3. *Count the number of m-simplexes needed to construct an n-simplex. An example is provided.*

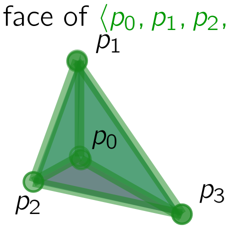
# of m-simplexes contained in an n-simplex	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$		0	0	0	0
$n=1$			0	0	0
$n=2$	3	3	1	0	0
$n=3$					
$n=4$					

Question 1.4 (Challenge question)**.** Can you think of a general rule for these formulas?

Example 1.5. $\langle p_1, p_3\rangle$ is a 1-*dimensional face* of $\langle p_0, p_1, p_2, p_3\rangle$:



Similarly, $\langle p_0, p_3, p_2\rangle$ is a 2-dimensional face of $\langle p_0, p_1, p_2, p_3\rangle$:



2. A SIMPLICIAL COMPLEX

Definition 2.1 (simplicial complex)**.** A simplicial complex K is a collection of simplices such that

(1) If K contains a simplex σ, then K also contains every face of σ.
(2) If two simplices in K intersect, then their intersection is a face of each of them.

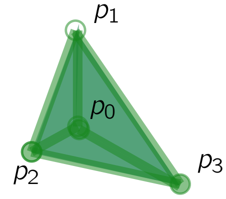
Remark 2.2. When we write a simplex K, we use *set notation* (that is, squiggly brackets containing all of the simplexes which are included in the simplicial complex: $K = \{\sigma_1, \ldots, \sigma_n\}$).

In comparison to a *simplex*, we think about a simplicial complex as a set with a visual representation. The simplex is a building block to create the simplicial complex.

Example 2.3. Here are some examples (and a nonexample) of a simplex, including both the *diagram and set notation*.

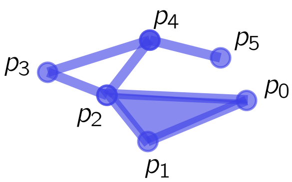
$\boxed{\text{Ex 1}}$

$$K = \begin{cases} \langle p_0, p_1, p_2, p_3\rangle, \langle p_1, p_2, p_3\rangle, \langle p_0, p_2, p_3\rangle, \langle p_0, p_1, p_3\rangle, \langle p_0, p_1, p_2\rangle, \\ \langle p_0, p_1\rangle, \langle p_0, p_2\rangle, \langle p_0, p_3\rangle, \langle p_1, p_2\rangle, \langle p_1, p_3\rangle, \langle p_2, p_3\rangle, \langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle. \end{cases}$$



$\boxed{\text{Ex 2}}$

$$K = \begin{cases} \langle p_0, p_1, p_2\rangle, \langle p_0, p_1\rangle, \langle p_0, p_2\rangle, \langle p_1, p_2\rangle, \langle p_2, p_3\rangle, \langle p_2, p_4\rangle, \langle p_3, p_4\rangle, \langle p_4, p_5\rangle, \\ \langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_5\rangle. \end{cases}$$

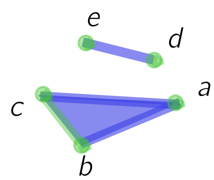


$\boxed{\text{Ex 3}}$ These are a few **NON-EXAMPLES**, which we will denote by J (they aren't simplicial complexes).

Task 4. *In the spaces below explain why these sets J are not simplicial complexes and how you would fix them (Draw J first, then add the appropriate simplex pieces to turn J into a simplicial complex K). For simplicity, we will use distinct letters from the alphabet to label the points. Check with your neighbor on what you drew- do these look correct?*

(**Example**) $J = \{\langle a, b, c\rangle, \langle a, b\rangle, \langle a, c\rangle, \langle d, e\rangle\}$

Solution:



$J \mapsto K$

$K = \{\langle a, b, c\rangle, \langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle, \langle d, e\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle\}$

Note that this 'corrected simplicial complex' K has two disjoint pieces. This is okay! It still satisfies the definition of a simplicial complex.

$J = \{\langle a, b, c\rangle, \langle b, c\rangle, \langle c, d\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle\}$

Solution:

$K =$

$J = \{\langle a, b, c, d\rangle, \langle d, e, f\rangle\}$

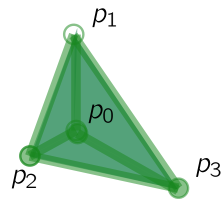
Solution:

$K =$

2.1. **Skeletons.** Now that we have some examples of simplexes, we are discuss how to sort and consider the various pieces which constitute the simplex. In particular, their 'skeletons!' .

Definition 2.4 (p-skeleton)**.** The *p-skeleton* of a simplicial complex K is denoted by $K^{(p)}$ and is the set of all of the simplices in K of dimension p or less.

Example 2.5. We give lists of the skeletons corresponding to Ex 1 and Ex 2 above. Fill in the gaps.



$\boxed{\text{Ex 1}}$

$K^{(0)} = \{\langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle\}$ (the vertices)

$K^{(1)} = \{\underline{\hspace{12cm}}\}$

(the vertices and the edges)

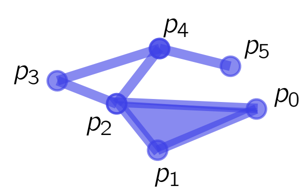
$K^{(2)} = \{\langle p_0, p_1, p_2, p_3 \rangle, \langle p_1, p_2, p_3 \rangle, \langle p_0, p_2, p_3 \rangle, \langle p_0, p_1, p_3 \rangle, \langle p_0, p_1, p_2 \rangle,$
$\langle p_0, p_1 \rangle, \langle p_0, p_2 \rangle, \langle p_0, p_3 \rangle, \langle p_1, p_2 \rangle, \langle p_1, p_3 \rangle, \langle p_2, p_3 \rangle, \langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle\}$

(the vertices, the edges and the triangles)

$K^{(3)} = K$ (the vertices, the edges, the triangles and the tetrahedron).

Task 5. *Draw $K^{(n)}$ for each n listed. Does the name 'skeleton' make sense?*

$K^{(0)}$ $K^{(1)}$ $K^{(2)}$ $K^{(3)}$



$\boxed{\text{Ex 2}}$ List the elements in each set:

$K^{(0)} = \{\underline{\hspace{12cm}}\}$

$K^{(1)} = \{\underline{\hspace{12cm}}\}$

$K^{(2)} = \{\underline{\hspace{12cm}}\}$

Task 6. *Draw $K^{(n)}$ for each n listed.*

$K^{(0)}$ $K^{(1)}$ $K^{(2)}$ $K^{(3)}$

Question 2.6. Argue whether each of these claims is true or false.

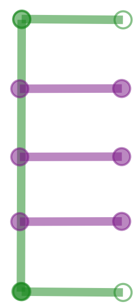
- For all n, $K^{(n)} \subset K^{(n+1)}$.
- If $n = \dim(K)$, $K^{(n)} = K$.
- If $n > \dim(K)$, $K^{(n)} = \emptyset$ (the empty set).

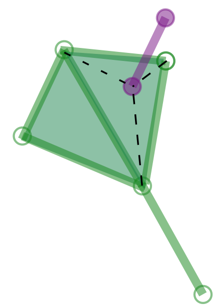
2.2. **Model complexes.**

Task 7. *Create simplicial complexes which model the following real life objects, including both a (1) labeled diagram and (2) set notation, as we did in examples above.*

Object	Diagram	Set notation
chair 		
bottle 		
balloon 		

Question 2.7 (**Challenge Question**)**.** How many faces would you need for a comb with n bristles? Or a brush with n bristles? (Here we have only added a few bristles to the comb and one to the brush, these are colored in purple)

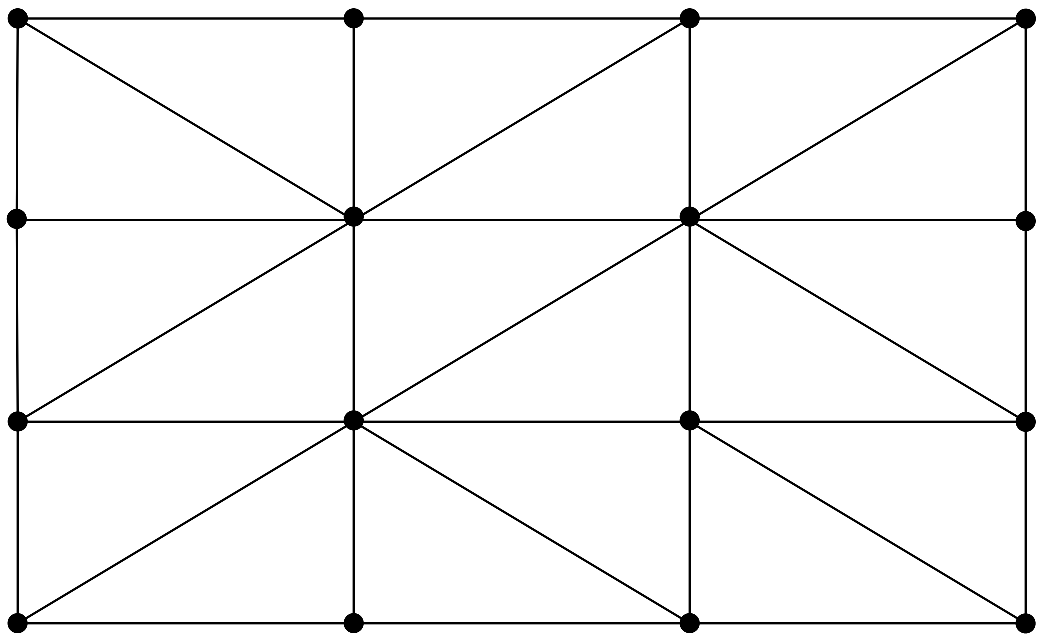


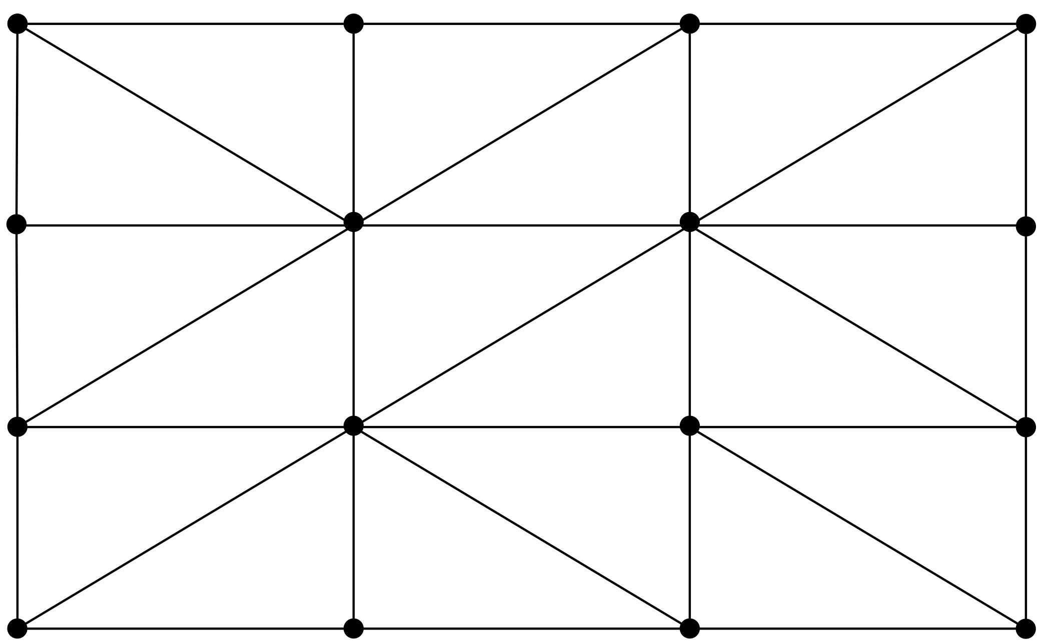


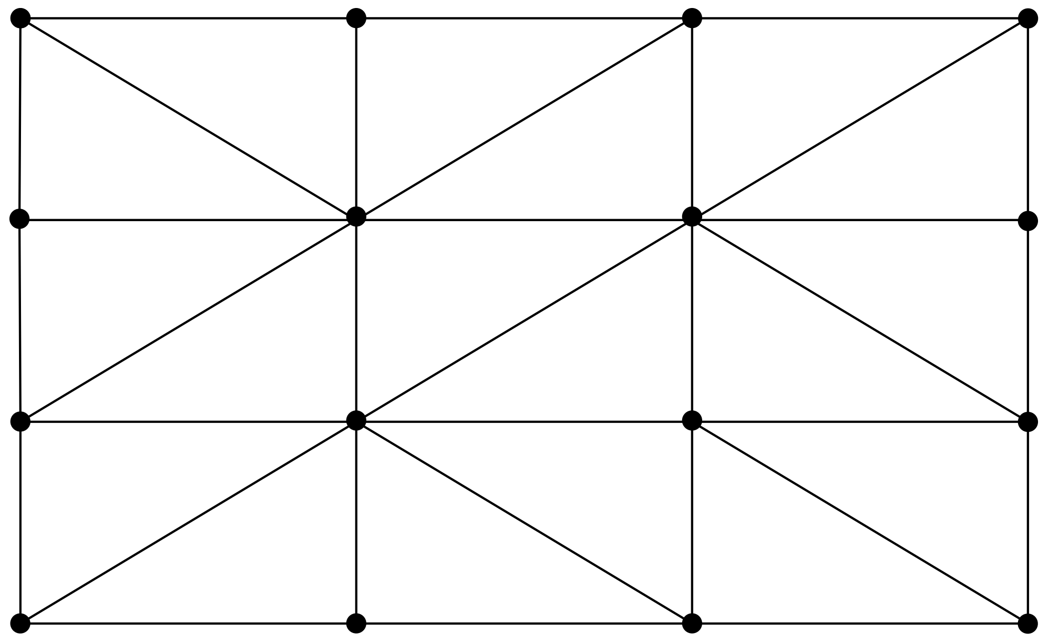

3. GLUING

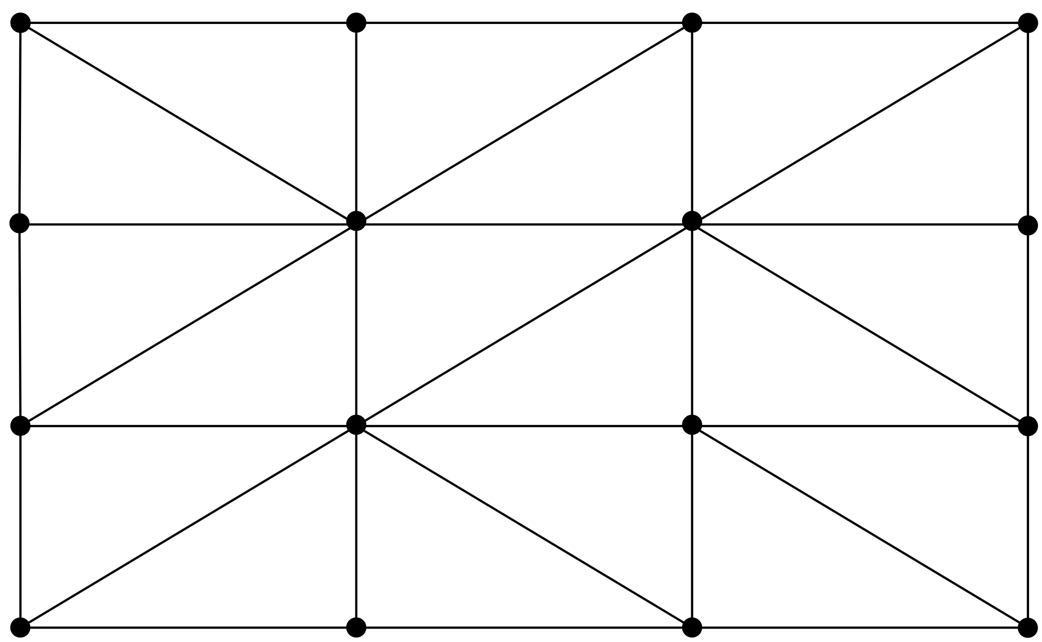
Task 8. *Using the grids on the next page, create the following objects*

(1) *Cylinder*
(2) *Möbius strip*
(3) *Torus*

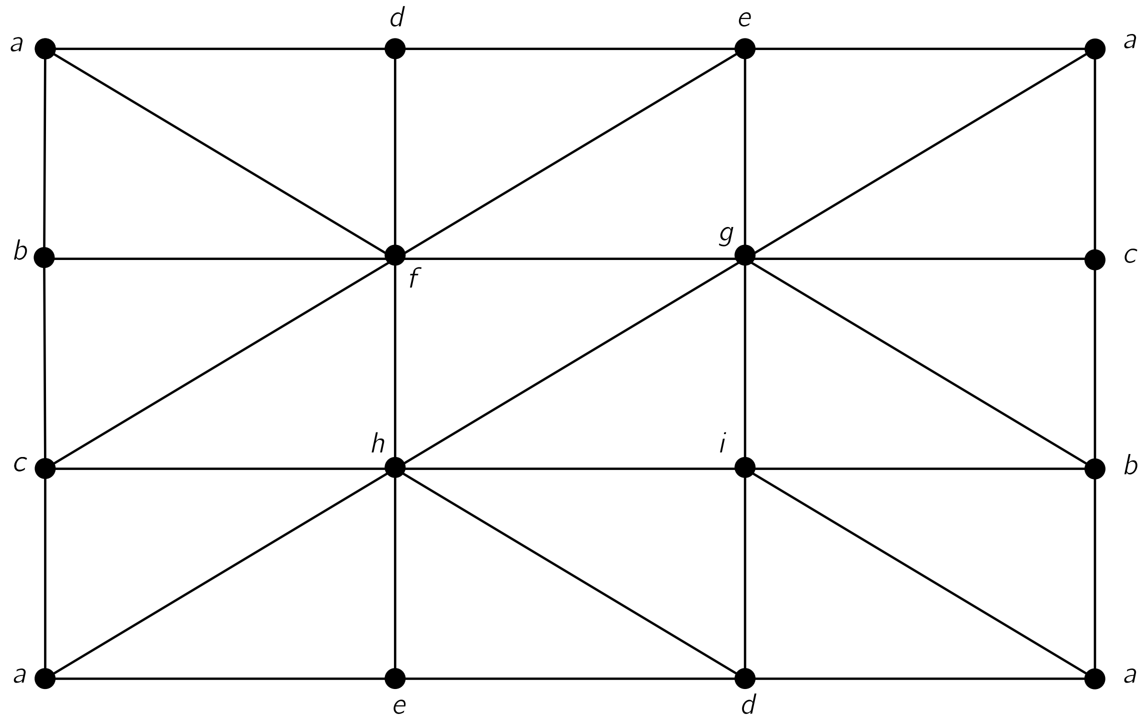








Task 9 (Challenge question)**.** *The structure you will get from this is a four dimensional surface is the* Klein bottle*. Using this grid, draw a sketch of how you think this will look (you can draw it with sides intersecting).*



Sketch:

ORIENTATION

Now we are going to add orientation to our simplicial complex. It may seem strange at first, but it will help later with both gluing and appropriately defining out boundary operator.

Definition 3.1 (oriented simplex)**.** $\vec{\sigma}$ is an *oriented p-simplex* if it is a p-simplex and has a fixed orientation (that is, the order of the vertexes is fixed). To denote an *oriented* simplex, we will use brackets $[\cdot]$ instead of $\langle\cdot\rangle$ symbols around the generating vertices.

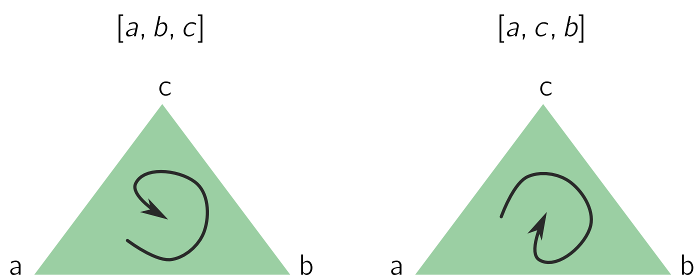
These oriented simplices come with the property that

$$\left[p_0, p_1, \dots, p_i, \dots, p_j, \dots p_{n-1}, p_n\right] = -\left[p_0, p_1, \dots, p_j, \dots, p_i, \dots p_{n-1}, p_n\right].$$

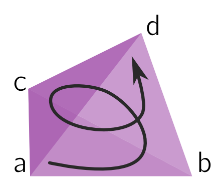
(Switching two vertices introduces a minus sign)

To draw oriented simplices, we will only consider n-simplices for $n \in [1, 2, 3]$

Example 3.2. Here are pictures of 1-oriented simplices, and 2-oriented simplices.



Finally, for a 3 simplex, we would draw $[a, b, c, d]$ as follows.



Task 10. *There are 6 orderings of the vertices of a triangle. However, there are only 2 oriented 2-simplices:* $[a, c, b]$ *and* $[a, b, c]$*. For example:*

$$[c, a, b] = -[a, c, b] = [a, b, c].$$

Similarly, a tetrahedron would only have two orientations: $[a, c, b, d]$ *and* $[a, b, c, d]$*. Which of these two is equivalent to* $[c, d, a, b]$*?*

Definition 3.3 (p-chain)**.** We can 'add' p-simplices with integer coefficients to form a *chain*.

Remark 3.4. In a way, you can think or an oriented simplex as representing an action. $[a, b]$ represents moving from a to b, so if you move from a to b and then from b to a, then the 2-chain representing this is the same as 'adding' the movements:

$$[a, b] + [b, a] = [a, b] - [a, b] = 0.$$

Which is saying, mathematically, 'you didn't get anywhere'!

Similarly, 2-simplices are like 'turns' in different directions, so adding $[a, b, c]$ to $[a, c, b]$ also vanishes. However, these keep track of how many 'total' spins made (so even if you add up many turns in the same direction they never cancel out each other).

Definition 3.5 (Oriented simplicial complex)**.** An *oriented simplicial complex* is a simplicial complex where all of its chains are oriented. If a simplicial complex K has oriented subsimplices, it is an oriented simplicial complex and denoted by $\vec{K}$.

Example 3.6. Here are some oriented simplicial complexes. Note that we have taken our nonoriented examples from before and put an orientation on them.

$$\vec{K} = \begin{cases} [p_0, p_1, p_2], [p_0, p_1], [p_0, p_2], [p_1, p_2], [p_2, p_3], [p_2, p_4], [p_3, p_4], [p_4, p_5], \\ [p_0], [p_1], [p_2], [p_3], [p_4], [p_5]. \end{cases}$$

A key point is *it doesn't* matter which way you orient each subsimplex, as long as you orient them! Now remember, the direction that you orient them does effect what they are equal to.

For example, (we've <u>underlined</u> the change):

$$\vec{K}' = \begin{cases} [p_0, p_1, p_2], \underline{[p_1, p_0]}, [p_0, p_2], [p_1, p_2], [p_2, p_3], [p_2, p_4], [p_3, p_4], [p_4, p_5], \\ [p_0], [p_1], [p_2], [p_3], [p_4], [p_5]. \end{cases}$$

As only simplicial complexes, K is equal to K', but as *oriented* simplicial complexes they are <u>not</u> equal.

Definition 3.7 (Simplicial complex chains)**.** The set of all possible p-chains generated from an oriented simplicial complex $\vec{K}$ is denoted by $C_p\left(\vec{K}\right)$.

Example 3.8. Here are some examples of p-chains in K:

- **(1-chain)**: $c_1 = [p_0, p_1] + 3[p_2, p_3]$,
- **(1-chain)**: $c_1 = -2[p_2, p_4] + 3[p_0, p_2]$,
- **(2-chain)**: $c_2 = 4[p_0, p_1, p_2]$.

Question 3.9. For the two examples above, how does $C_p\left(\vec{K}\right)$ compare to $C_p\left(\vec{K}'\right)$? (Hint: how does a negative sign help?). Can you say something in general about oriented simplicial complexes formed from the same simplicial complexes?

BOUNDARY

Now we can finally define the boundary operator.

Definition 3.10 (Boundary operator)**.** ∂ acts on oriented p-simplices as follows

$$\partial\left[p_0,\ldots,p_n\right]=\sum_{i=1}^{n}(-1)^n\left[p_0,\ldots,\hat{p}_i,\ldots,p_n\right],$$
$$\partial\left[p_i\right]=0.$$

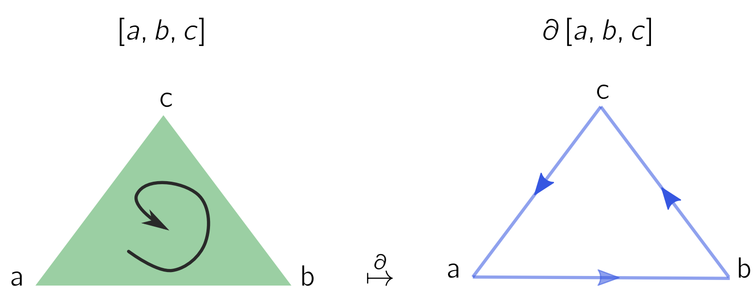
(Here the hat $\vec{p}_i$ means we take p_i out from the simplex.) The main point is we go through writing the $(p-1)$-simplices where the ith entry is taken out, but with alternating signs.

Since this definition can be confusing, let's pause to give an example.

Example 3.11.

$$\begin{aligned}\partial\left[a,b,c\right]&=(-1)^0\left[\hat{a},b,c\right]+(-1)^1\left[a,\hat{b},c\right]+(-1)^2\left[a,b,\hat{c}\right]\\&=\left[b,c\right]-\left[a,c\right]+\left[a,b\right]\\&=\left[b,c\right]+\left[c,a\right]+\left[a,b\right]\end{aligned}$$

Here is a visual, you can visually see how it literally computed the boundary of the shape!



Task 11. *Compute $\partial\left[a,b,c,d\right]$ using the formula, then sketch your results as we did in the example (use the space in the next page).*

Definition 3.12. To extend ∂ to a p-chain, we just make it 'hit' all of the simplices and ignore signs and constants in between.

$$\partial\left(c\vec{\sigma}\right)=c\partial\vec{\sigma},\qquad \text{where } c \text{ is a scalar, and } \sigma \text{ is a simplex}$$
$$\partial\left(\vec{\sigma_1}+\vec{\sigma_2}\right)=\partial\vec{\sigma}_1+\partial\vec{\sigma}_2,\qquad \text{where } \sigma_1,\sigma_2 \text{ are two simplices.}$$

Example 3.13 (Boundary of a chain)**.** Consider the *bowtie* below, given by

$$\vec{c}=\left[a,b,c\right]+\left[c,d,e\right]$$

if we compute the boundary we have

$$\begin{aligned}\partial\left(\left[a,b,c\right]+\left[c,d,e\right]\right)&=\partial\left[a,b,c\right]+\partial\left[c,d,e\right]\\&=\left[b,c\right]-\left[a,c\right]+\left[a,b\right]+\left[d,e\right]-\left[c,e\right]+\left[d,e\right].\end{aligned}$$

Now if we compute ∂ again using what we computed above, we have

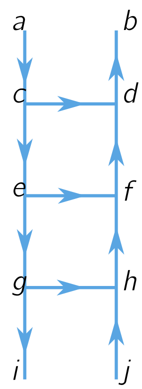
$$\begin{aligned}
\partial^2\left([a,b,c]+[c,d,e]\right) &= \partial\left(\partial\left([a,b,c]+[c,d,e]\right)\right)\\
&= \partial\left([b,c]-[a,c]+[a,b]+[d,e]-[c,e]+[d,e]\right)\\
&= \partial\,[b,c]-\partial\,[a,c]+\partial\,[a,b]+\partial\,[d,e]-\partial\,[c,e]+\partial\,[c,d]\\
&= (b-c)-(a-c)+(a-b)+(d-e)-(c-e)+(c-d)\\
&= 0.
\end{aligned}$$

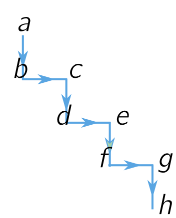
Task 12. *Compute ∂^2 (that is, operate ∂ twice) on the following simplicial complexes.*

(1) $\sigma = [a,b]+[b,c]$,
(2) $\sigma = [a,b,c]$ *(use the work from Example 3.11)*,
(3) $\sigma = [a,b,c,d]$ *(use your work from Task 11)*.

In fact, we always have that for every chain σ, we have that $\partial^2\sigma \equiv 0$.

Task 13. *Now that you have computed a sufficient amount of boundaries, try to use your intuition to determine the coefficients of the vertices in the boundary of the following 1-chains:*





SIMPLICIAL COLLAPSE

Definition 3.14 (maximal element)**.** Let K be a simplicial complex. A face a of K is called *a maximal element of* K if it is not a face of any simplex of K, except itself.

The simplicial complex pictured below has 5 maximal elements: the tetrahedron $\langle B, E, F, G\rangle$, the triangle $\langle A, E, H\rangle$ and the three segments $\langle B, C\rangle$, $\langle C, D\rangle$ and $\langle B, D\rangle$. The triangle $\langle B, G, E\rangle$ is not maximal because it is a face of the tetrahedron.

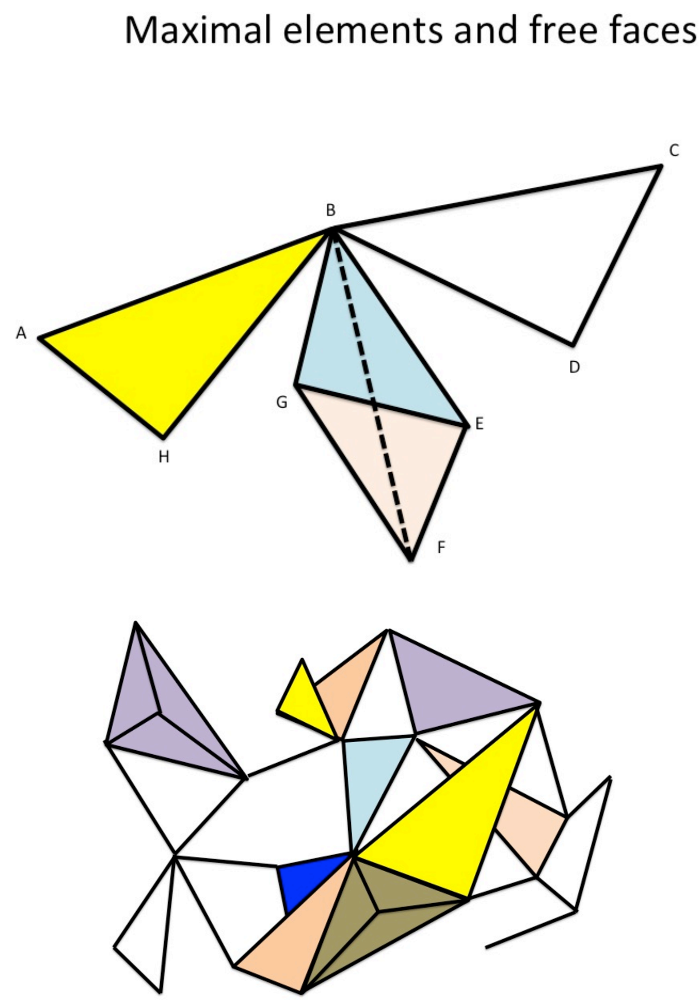
Definition 3.15 (free face)**.** If a is a maximal element of a simplicial complex K, a face b of a is called *a free face* if $b \neq a$ and b is not contained in any other simplex of K.

In our example, the segment $\langle B, C\rangle$ has no free faces, while the triangle $\langle A, E, H\rangle$ has three free faces (its edges).

Task 14. *In the simplicial complex K pictured below, identify six different pairs (a, b) where a is a maximal element of K and b is a free face of a.*

Question 3.16 (**Challenge Question**)**.** *True or false?* If a is a free face of a maximal element b, then a has codimension 1. That is, the dimension is the simplex a is 1 less than the dimension of the simplex b.

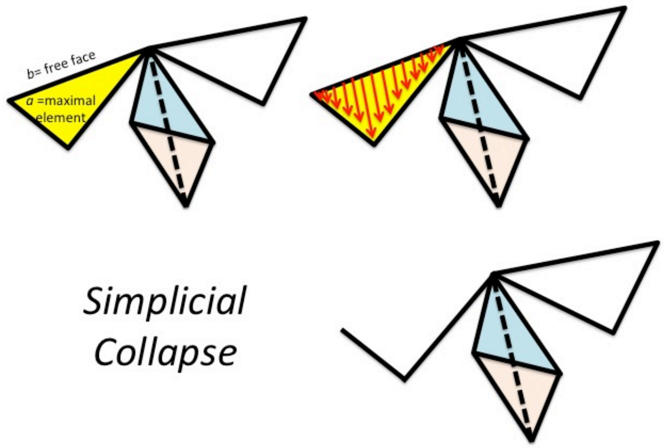
Definition 3.17 (simplicial collapse)**.** Let K be a simplicial complex. Suppose that a is a maximal element of K and b is a free face of a. The act of removing the faces $\{a, b\}$, replacing K by the simplicial complex $K - \{a, b\}$ is called *simplicial collapse*.



You can think of a simplicial collapse as a removal of maximal element of K and its free face, by *pushing in* the free face, until the entire maximal simplex disappears. Note: all the remaining faces of the maximal element remain.

Several examples of simplical collapses are shown in the pictures below. We iterate the construction until no more free faces are found (thus reducing the original simplicial complex to its bone structure.)

Task 15. *Collapse the open book, until there are no free faces left.*


b= free face
a =maximal element
Simplicial Collapse