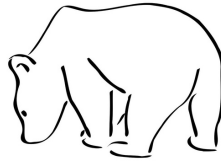


TRIANGLES IN DIFFERENT GEOMETRIES

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Today we are going to look at angles on different types of surfaces and relate it to the properties on these objects.



We start off with a riddle...

A bear leaves home and walks 100 miles south, rests, and then turn west and walks straight for 100 miles. She then turns once more and walks north, and arrives home. What color is the bear? (Hint: Save this question for later, after you have worked through some of the activities).

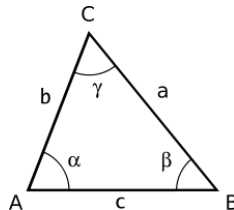
(Riddle answer):

1. Introduction

First, just to recall, we give the definition of 'triangle' generally which we will use for our purposes. We will see what the general definition is important.

Definition 1.1 (Triangle). Let S be a surface. A *triangle* on S is the region created by connecting three noncolinear points (all do not lie on the same line) with straight line segments.

Note that when constructing this, there will be three corresponding angles formed at each point (hence the name **tri-angle**).



We will let Σ represent the **sum of angles**, that is, with a setup like above, set

$$\Sigma = \alpha + \beta + \gamma.$$

2. Planar setting

Take out a blank sheet of paper and pen. This is the plane you will be working on when looking at the following activities. Think about the riddle above...

Question 2.1. If the bear was walking on a planar surface and made these turns would it have possibly reached home?

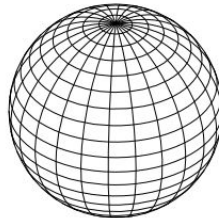
Question 2.2. What is the Σ for any triangle drawn on the plane?

3. Sphere setting

Recall the definition of triangle from Definition 1.1 while doing these computations.

Definition 3.1. A *great circle* is a circle on the surface of a sphere that lies in any plane which passes through the sphere's center. Its radius is the same as the radius of the whole sphere.

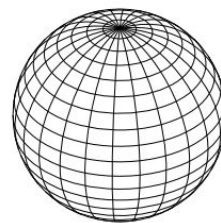
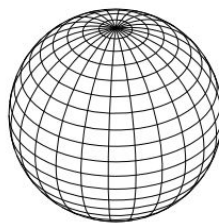
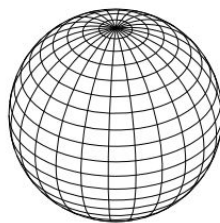
Task 1. Locate a great circle (or a few) on this diagram:

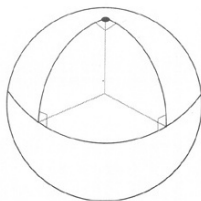


A line segment must lie along a great circle, since great circles trace out the shortest path from one point to another!

Definition 3.2. A *spherical triangle* is a triangle with sides that all lie along great circles.

Task 2. Sketch three examples of spherical triangles.





Task 3. Take three points on a sphere and connect them with straight lines over the surface of the sphere, to get the following spherical triangle with three angles of 90° .

Question 3.3. How many of these types of $90^\circ - 90^\circ - 90^\circ$ triangles exist on the sphere? What if you fix one point? What about two points?

Question 3.4. Do you think that all the triangles drawn on a sphere have angles which add up to $\Sigma = 270^\circ$?

Guess:

Task 4. Draw any triangle on the sphere, which we will call **Triangle 1** (make it fairly big!). Measure its angles and record them. Then draw another triangle inside of that (**Triangle 2**), and record its angles. Draw another triangle inside of Triangle 2, which we will call **Triangle 3** and measure its angles. Ask your neighbors for their data and record it here.

	Your angles	Neighbor 1's angles	Neighbor 2's angles
Triangle 1 :			
Triangle 2 :			
Triangle 3 :			

Question 3.5. Now, with this data, what can you say about the sums of the angles, Σ ?

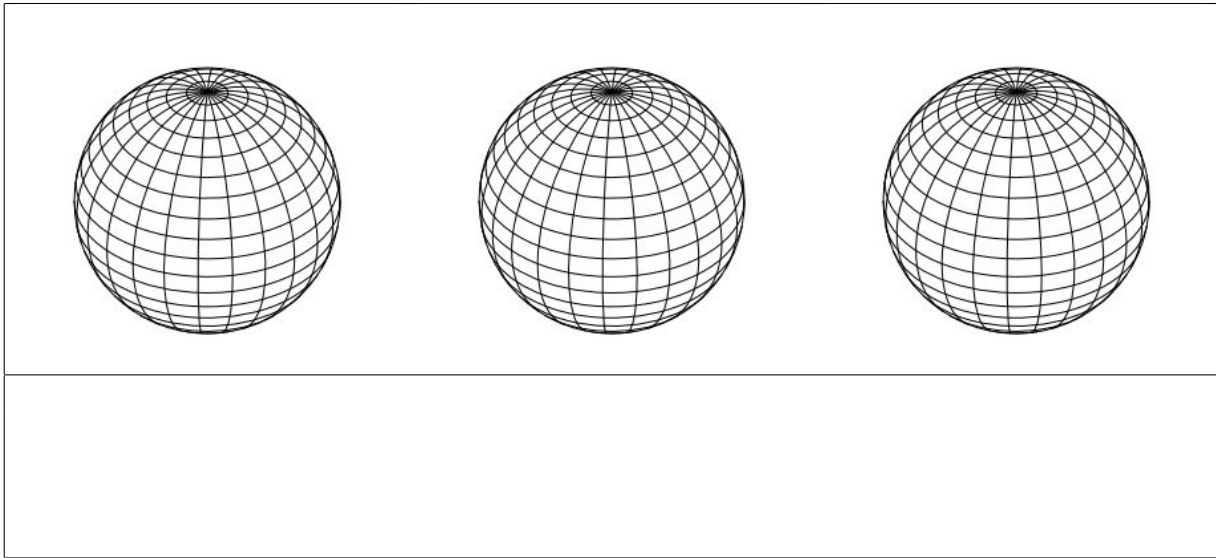
Conclusion:

Question 3.6. As the triangles get smaller and smaller, what do you think Σ limits to?

Conclusion:

Does your answer make sense? (Think about us living on the globe and how the world looks to us.)

Task 5. Now try to make a triangle with the largest sum of angles that you possibly can. Sketch your solution(s) on the spheres below, or describe it in the space below the spheres



Question 3.7. What is the largest vertex angle possible for a given triangle?

Question 3.8. What is a sharp upper bound for the sum of angles on a sphere? ('Sharp upper bound' means the best bound you can get). (Hint: you may not be able to get an actual triangle (that it satisfies Definition 1.1) but you can get as close as you want can to this value.

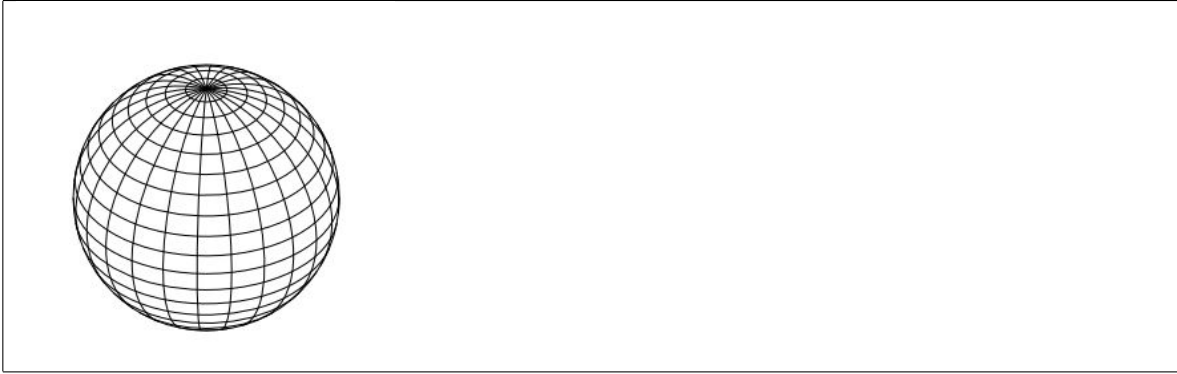
Question 3.9. Does a triangle need to necessarily be inside of only one hemisphere to fit the definition? Why or why not?

3.1. **Euclid's postulates.** Here we are going to describe *Euclid's postulates*, the following set of rules which are followed in the *planar* setting.

- (P1) A straight line segment can be drawn joining any two points.
- (P2) Any straight line segment can be extended indefinitely in a straight line.
- (P3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- (P4) All right angles are congruent.
- (P5) (Parallel postulate) If two line are drawn, and intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles (180°), then the two lines inevitably must intersect each other on that side far enough out.

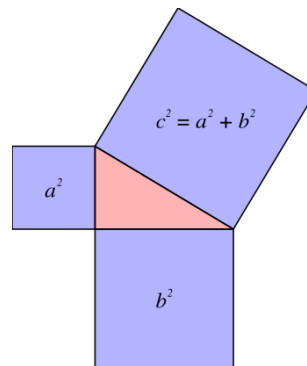
Task 6. Draw some pictures depicting (P5) in action.

Question 3.10. Which one of these postulates fail in the spherical setting? Why? Draw an example of what could go wrong, and write some notes explaining why.



3.2. **Spherical Pythagorean Theorem.** Do you know or remember the Euclidean Pythagorean Theorem? We'll state it here.

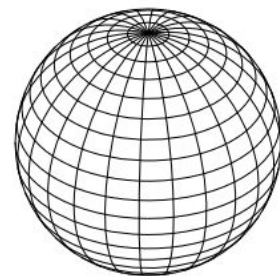
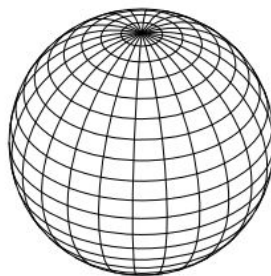
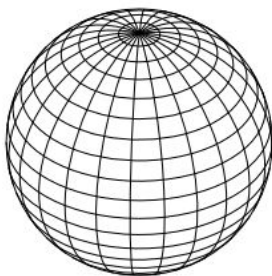
Theorem 3.11 (Pythagorean Theorem). *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*



Our goal is to derive an equivalent theorem for the spherical case. To start, we have to define a particular kind of triangle.

Definition 3.12. A *spherical right triangle* is a triangle with sides along great circles, which has one angle that is 90° .

Task 7. Sketch three examples of spherical right triangles. Make one of them an equilateral right triangle. What is special about this particular triangle? (think about this kind of triangle in the planar setting).



We will let the *spherical hypotenuse* be the side of the triangle opposite of the right angle, and denote it by c . Label the other sides a and b . Let the radius of the sphere be R .

Theorem 3.13 (Spherical Pythagorean Theorem). *Using the labelling above we have the following.*

$$\left(\cos \frac{c}{R}\right) = \left(\cos \frac{a}{R}\right) \left(\cos \frac{b}{R}\right).$$

Here 'cos' denotes the cosine function. If you haven't seen it, for now, we can just plug it into our calculators.

Task 8. When your 'spheres' are blown up to full size their radius is approximately _____. Using this, we want you to verify the Pythagorean Theorem (it may be off by a little, since these are approximations). To do so, make three right triangles, and fill in the table.

	a	b	c	$\left(\cos \frac{a}{R}\right)$	$\left(\cos \frac{b}{R}\right)$	$\left(\cos \frac{a}{R}\right) \left(\cos \frac{b}{R}\right)$	$\left(\cos \frac{c}{R}\right)$
Triangle 1 :							
Triangle 2 :							
Triangle 3 :							

Now we are going to show together that you can obtain the planar Pythagorean theorem from the spherical one. To do so, we need the following interesting fact about cosine.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{2n}} (2n)! = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

For the numbers we are dealing with we only need the first few terms on the right hand side, so we can take

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

Task 9. Plug in $x = \frac{c}{R}$ into the equation for cosine (replace every x with $\frac{c}{R}$), and then simplify it:

$\left(\cos \frac{c}{R}\right) =$

Task 10. Now do the same, for the following

$\left(\cos \frac{a}{R}\right) \left(\cos \frac{b}{R}\right) =$

$$\left(\cos \frac{c}{R}\right) = \left(\cos \frac{a}{R}\right) \left(\cos \frac{b}{R}\right) \text{ expanded out...}$$

Task 11. Now finally combine what you wrote.

Together as a group, we will simplify this expression.

This last step we do is 'blowing up the sphere'. As the sphere gets bigger and bigger it looks flatter and flatter (like how we feel like we are basically living on a flat surface). We need to see how all the terms behave as R gets big, that is, $R \rightarrow \infty$. First we have that

$$\frac{1}{R} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

To check this you can plug in larger and larger values for R to see how it is behaving:

R	$\frac{1}{R}$
1	1
100	$\frac{1}{100}$
100,000	$\frac{1}{100,000}$
100,000,000	$\frac{1}{100,000,000}$

The right hand side is getting very very small... and approaching 0.

Task 12 (The conclusion!). Look at how your expression in Task 11 behaves as you take $R \rightarrow \infty$. What do you get?

4. Conclusions and Extensions

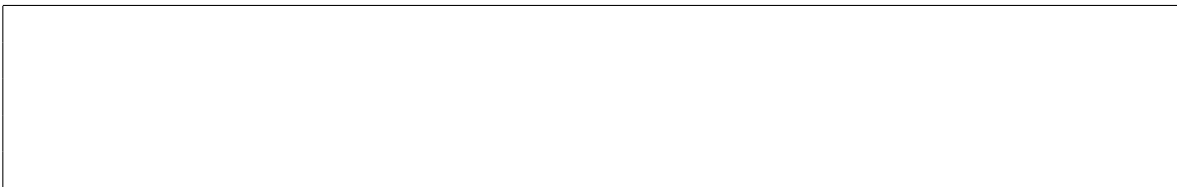
There are actually three types of geometries: **Euclidean**, **Spherical** and **Hyperbolic**. If you haven't guessed, **(P5)** fails in the spherical case, and also in the hyperbolic case.

- **(P5, spherical)**: For a given line L , if we take a point P not on the line, **NONE** of the lines going through P can avoid intersecting L .
- **(P5, hyperbolic)**: For a given line L , if we take a point not on the line, there are **INFINITELY** many lines going through P which avoid intersecting L .

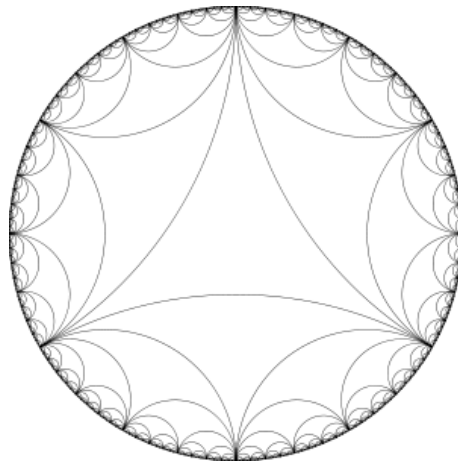
Task 13. We will draw 'interpretations' of these at the board. Record these onto your worksheet.



Task 14. Do these both make sense? Try to disprove **(P5, spherical)**. Sketch possible counter examples, and discuss with your neighbors what could go wrong.



What about hyperbolic space? Well, it's a little hard to study mathematically. We can use something like the Poincaré disk to help us work on it. Here some straight lines (in hyperbolic space) are marked on the disk. The disk model *preserves angles*, but the sizes of objects are distorted.



Task 15. *Based on the diagram, what do you think about the angles of triangles in this space versus on the plane or sphere? What do you think is the sharp upper bound of Σ here? What about the sharp lower bound?*

Even though it's hard to draw, you can see hyperbolic geometry in nature. On these surfaces it is possible to 'draw' triangles as we have been doing in the spherical case.



clam



lettuce



flatworm

Question 4.1. Can you think of other things that have this 'ruffled' hyperbolic types of shapes? (A lot of underwater objects actually exhibit this structure.)

Interestingly, one of the best ways to model hyperbolic space is using crochet!

