Minimum Number of Experiments

Geoff Galgon, Garrett Ervin

Abstract

We give some examples of puzzles which are of the general form "how few experiments are required to determine the unknown?" We also discuss a general heuristic for solving them.

Contents

1 Questions and Doors

Question 1.1. You're on a game show and are presented with three doors. Behind one of the doors is a prize, but you don't know which one. If you're allowed to ask exactly one yes-or-no question to the show's host (who always responds honestly), is it possible to determine with certainty which door the prize is behind?

Question 1.2. You're on a game show and are presented with three doors. Behind one of the doors is a prize, but you don't know which one. This time you're allowed to ask *two* yes-or-no questions. Is it possible to determine with certainty which door the prize is behind?

Question 1.3. You're on a game show and are presented with four doors. Behind one of the doors is a prize, but you don't know which one. Again you're granted two yes-or-no questions. Is it possible to determine with certainty which door the prize is behind?

Question 1.4. Generally speaking, if there are n doors, what is the minimum number of yes-or-no questions you have to ask in order to unambiguously determine which door the prize is behind? Which questions are they?

Question 1.5. You're on the show again, and it's back down to four doors. You're allowed to ask two yes-or-no questions, but this time you have to ask both questions before either answer is given. Questions that refer to the other question are not allowed (for example, you cannot begin a question with "If the answer to the other question is yes..."). Can you determine which door the prize is behind? How many questions do you need if there are 8 doors? Or n doors?

Question 1.6. The rules of the game have changed. Now you can ask not only yes-or-no questions, but *any* question that has at most two possible answers. For example, "Is the prize to the right of where I'm standing, or to the left?" is now a legal question since there are only two possible answers ("right" and "left"), whereas "Is the prize behind door 1, door 2, or otherwise?" is not legal, since there are three possible answers ("door 1," "door 2," and "otherwise"). Now how many questions do you need to find the prize if there are four doors? Or n doors?

Question 1.7. You can now ask questions with at most *three* possible answers. How many such questions do you need to find the prize if there are four doors? Or nine doors? Or n doors?

2 Weighing Coins

Question 2.1. You have four coins, one of which is counterfeit. You don't know if it's heavier or lighter. You also have a fifth coin that you know is real. What is the minimum number of weighings required to determine which coin is counterfeit, and whether it is heavier or lighter. This is a balance scale.

Question 2.2. You have nine coins, one of which is counterfeit, and you know it's lighter than the real coins. What is the minimum number of weighings required to determine which is the counterfeit coin?

Question 2.3. You have twelve coins, one of which is counterfeit, and you don't know whether it's heavier or lighter. What is the minimum number of weighings required to determine which is the counterfeit coin (and whether it is heavier or lighter)?

Question 2.4. You have thirteen coins, one of which is counterfeit, and you don't know whether it's heavier or lighter. What is the minimum number of weighings required to determine which is the counterfeit coin (you do not need to determine if the coin is heavier or lighter)?

Question 2.5. Can you think of a way of solving the previous coin problem(s) in a way that is independent of experiment order (like Question 1.5)?

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We give some examples and solution sketches of puzzles which are of the general form "how few experiments are required to determine the unknown?" We also discuss a general heuristic for solving them.

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1 Questions and Doors

Question 1.1. You're on a game show and are presented with three doors. Behind one of the doors is a prize, but you don't know which one. If you're allowed to ask exactly one yes-or-no question to the show's host (who always responds honestly), is it possible to determine with certainty which door the prize is behind?

Solution: A general idea we're trying to communicate with these questions is that any solution will necessitate (in particular) the existence of a surjective function from the space of sequential experimental outcomes to the appropriate state space. In this case, there are only two possible experimental outcomes, while the state space has size three, so this is impossible.

Question 1.2. You're on a game show and are presented with three doors. Behind one of the doors is a prize, but you don't know which one. This time you're allowed to ask *two* yes-or-no questions. Is it possible to determine with certainty which door the prize is behind?

Solution: The difference here is that the experimental space has size four, call it e.g. $\{00, 01, 10, 11\}$, and the state space has size three, so the existence of a solution isn't ruled out. An example of something that works would be "is it in those two doors?" followed (if necessary) by "is it in the one on the left, of those two"?

Question 1.3. You're on a game show and are presented with four doors. Behind one of the doors is a prize, but you don't know which one. Again you're granted two yes-or-no questions. Is it possible to determine with certainty which door the prize is behind?

Solution: $|\text{state space}| = |\text{experimental space}|$, and in this case a solution can be given by, for example, "is it in those two doors?" followed (it is necessary this time) by "is it in the one on the left, of those two"?

Question 1.4. Generally speaking, if there are n doors, what is the minimum number of yes-or-no questions you have to ask in order to unambiguously determine which door the prize is behind? Which questions are they?

Solution: By progressive bisections, the minimum number of yes-or-no questions will be the smallest integer greater than or equal to $\log_2(n)$.

Question 1.5. You're on the show again, and it's back down to four doors. You're allowed to ask two yes-or-no questions, but this time you have to ask both questions before either answer is given. Questions that refer to the other question are not allowed (for example, you cannot begin a question with "If the answer to the other question is yes..."). Can you determine which door the prize is behind? How many questions do you need if there are 8 doors? Or n doors?

Solution: While it may be most obvious to think of the bisection strategy as above, taking the general view that each experiment will partition the state space, according to the possible outcomes of the experiment, the task now becomes to ask two questions so that the intersection of the resulting partitions consists of the singletons. An example here would be "is it in $\{1, 2\}$?" and "is it in $\{1, 3\}$?" For 8 doors, the questions could be "is it in $\{1, 2, 3, 4\}$?", "is it in $\{1, 2, 5, 6\}$?", and "is it in $\{1, 3, 5, 7\}$?" One can proceed generally.

Question 1.6. The rules of the game have changed. Now you can ask not only yes-or-no questions, but *any* question that has at most two possible answers. For example, "Is the prize to the right of where I'm standing, or to the left?" is now a legal question since there are only two possible answers ("right" and "left"), whereas "Is the prize behind door 1, door 2, or otherwise?" is not legal, since there are three possible answers ("door 1," "door 2," and "otherwise"). Now how many questions do you need to find the prize if there are four doors? Or n doors?

Solution: This question is intended to emphasize that questions with only two answers will still yield experimental spaces of the same size (yes or no could have been 0 or 1, etc.).

Question 1.7. You can now ask questions with at most *three* possible answers. How many such questions do you need to find the prize if there are four doors? Or nine doors? Or n doors?

Solution: If we're allowed three possible outcomes to every experiment, the possible experimental space is now of size 3^n instead of 2^n, where n is the number of experiments. So, for example, we can answer the nine door case in three questions of this type. One can perhaps think here of trying to gain a trit of information with each successive experiment as opposed to a bit of information as was the case previously.

2 Weighing Coins

Question 2.1. You have four coins, one of which is counterfeit. You don't know if it's heavier or lighter. You also have a fifth coin that you know is real. What is the minimum number of weighings required to determine which coin is counterfeit, and whether it is heavier or lighter. This is a balance scale.

Solution: We don't know if this counterfit coin is heavier or lighter, so the state space is of size 8. A key to these balance puzzles is that each experiment has three possible outcomes (right side heavier, left side heavier, and equal), so we're dealing with trits instead of bits, as in Question 1.7. In any case, if we have two experiments then our experiment space is of size $9 > 8$, so let's see if this works. First weigh $\{1, 2\}$ against $\{3, 5\}$. If these are different, then the counterfit coin is among $\{1, 2, 3\}$, so just weigh e.g. 1 against 2 and sort accordingly, if the first weighing the same, the counterfit is 4, and weigh again to determine if it is heavier or lighter.

Question 2.2. You have nine coins, one of which is counterfeit, and you know it's lighter than the real coins. What is the minimum number of weighings required to determine which is the counterfeit coin?

Solution: The state space is of size 9, because we know the counterfit coin is lighter. So we can try to search for a solution in two weighings. Indeed this works, for example by first weighing $\{1, 2, 3\}$ against $\{4, 5, 6\}$. If equal, then lighter coin is among $\{7, 8, 9\}$ and easy to determine in one weighing. If different, the lighter coin is in the lighter stack, and again easy to determine in one weighing.

Question 2.3. You have twelve coins, one of which is counterfeit, and you don't know whether it's heavier or lighter. What is the minimum number of weighings required to determine which is the counterfeit coin (and whether it is heavier or lighter)?

Solution: The state space is size 24, so we're going to need at least three experiments. It turns out we can solve this in three. One might try to either weigh groups of 4 or groups of three first. However, if we weigh two groups of three and they're the same, then we only know the counterfit is among a group of six, which we can't solve with two weighings, as the experimental space is of size 9 while the state space is of size 12. So we have to weigh two groups of four. In all cases one can see that the resulting state space will be of size 8, and the experimental space of size 9, a good sign. If the two groups are equal, it is not difficult to see that the counterfit coin as well as whether it's heavier or lighter can be determined from the remaining four coins, noting that we now know that the previously weighed 8 coins are all real.

So, without loss of generality assume $\{1, 2, 3, 4\}$ is heavier than $\{5, 6, 7, 8\}$. Next weigh, for example, $\{1, 2, 5\}$ vs. $\{3, 7, 12\}$. There are two cases, either they weigh the same, or they're different. Supposing they weigh the same, then the counterfit coin is among $\{4, 6, 8\}$, and can be determined as well as whether it's heavier or lighter by weighing 6 vs. 8. Supposing then that in the $\{1, 2, 5\}$ vs. $\{3, 7, 12\}$ weighing one is heavier, there are two subcases. Suppose first $\{1, 2, 5\}$ is heavier. We know then that the counterfit coin must be among $\{1, 2, 7\}$. Weighing 1 vs. 2. will yield the result, as well as whether it's heavier or lighter. Similarly, if $\{3, 7, 12\}$ is heavier, then the counterfit coin must be among $\{3, 5\}$, which is easy to determine in one weighing.

Question 2.4. You have thirteen coins, one of which is counterfeit, and you don't know whether it's heavier or lighter. What is the minimum number of weighings required to determine which is the counterfeit coin (you do not need to determine if the coin is heavier or lighter)?

Solution: The state space is of size 26, and the experimental space is of size 27, so there's hope for three weighings. It's interesting that one can't determine necessarily if the counterfit coin is heavier or lighter here, while one can in the previous case. Why? Any initial weighing will potentially yield a state space of size 10, which we couldn't determine with a subsequent experimental space of size 9. For example, if we weigh 4 vs. 4, then if they're equal the remaining state space has size 10 (as opposed to 8 in the previous problem).

However, having solved the 12 coin problem, just choose 12 coins of the 13 and proceed with the previous strategy. If all weighings are equal, (which wouldn't have happened before) then the 13th coin is counterfit (but we don't know whether it's heavier or lighter).

Question 2.5. Can you think of a way of solving the previous coin problem(s) in a way that is independent of experiment order (like Question 1.5)?

Solution: One can approach the problem of 12 coins without using the usual tree-like reasoning by thinking along the lines of Question 1.5. We have to find three experiments that partition the space of 24 possibilities into singletons. There are some added complications, but the following should work:

$$\begin{array}{c}\{3,4,6,9\} \text{ vs. } \{1,2,7,10\}\\ \{2,5,6,7\} \text{ vs. } \{1,3,8,11\}\\ \{1,4,5,8\} \text{ vs. } \{2,3,9,12\}\end{array}$$