Mathematical modeling of an epidemic

BY LAURE GIOVANGIGLI, POSTDOC AT UCI

1 The general epidemic model

1.1 Model assumptions

We divide the population into 3 groups

- Susceptible individuals who can catch the disease
- Infective individuals who can spread the disease
- Removed individuals who have recovered and are immune to the disease.

We denote by $S(t)$ the number of susceptible individuals at time t, $I(t)$ the number of infective individuals at time t and $R(t)$ the number of recovered individuals at time t.

We make the following assumptions

- The number of individuals in the population is large and constant equal to N. Thus for all t
  $$N = S(t) + I(t) + R(t).$$
  There is no birth, death, emigration or immigration.
- The population is well mixed.
- There is no latent period. Once you get the disease, you are immediately infective.
- The recovery rate γ is constant equal to $\frac{1}{\tau}$ where τ is the average duration of the infection.
- The infection rate β is constant equal to $c\chi$, where c is the number of contacts that one individual makes in the time unit and χ is the infectiveness of one contact with an infective.

1.2 Equations of change

If $S(n)$, $I(n)$ and $R(n)$ are the number of susceptibles, infective and removed individuals at time n, at time $n+1$ we have

$$
\begin{cases}
S(n+1) = S(n) - \beta \dfrac{I(n)}{N} S(n) \\
I(n+1) = I(n) + \beta \dfrac{I(n)}{N} S(n) - \gamma I(n) \\
R(n+1) = R(n) + \gamma I(n)
\end{cases}
\tag{1}
$$

Exercice 1 : Use the daily changes above to compute 4 days of a rubella epidemic. For rubella, we have that $\tau = 10$ and $\beta = 6$.

We suppose that a population has initially (at $t = 0$)

$$
\begin{aligned}
S(0) &= 10000 \\
I(0) &= 1000 \\
R(0) &= 19000
\end{aligned}
$$

(a) Find the total number of individuals in the population.

(b) Compute the recovery rate.

(c) Use the equations of change to complete a table showing the first four days of the epidemic.

t	0	1	2	3	4
S	10 000				
I	1 000				
R	19 000				

(d) Verify that $S + I + R = N$ at each time.

(e) Use your completed table to graph S versus t, I versus t and R versus t.

More generally, let Δt be a small time step. From $S(t)$ $I(t)$ and $R(t)$, we can compute $S(t + \Delta t)$, $I(t + \delta t)$ and $R(t + \Delta t)$ through the following equations

$$
\begin{cases}
S(t+\Delta t) = S(t) - \beta \dfrac{I(t)}{N} S(t) \Delta t \\
I(t+\Delta t) = I(t) + \beta \dfrac{I(t)}{N} S(t) \Delta t - \gamma I(t) \Delta t \\
R(t+\Delta t) = R(t) + \gamma I(t) \Delta t
\end{cases}
\tag{2}
$$

Exercice 2 : We go back to our example of the rubella.

(a) What does the following quantity measures and in what units?

$$
\left(\beta \frac{I(0)}{N} S(0) - \gamma I(0) \right) \cdot \frac{1}{2}.
$$

(b) Complete the table below

t	0	1/2	1	3/2	2
S	10 000				
I	1 000				
R	19 000				

In calculus, the limit of the rate of change as the time step goes to 0 is called the derivative and denoted as

$$\frac{dS}{dt} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t}$$

$$\frac{dI}{dt} = \lim_{\Delta t \to 0} \frac{\Delta I}{\Delta t} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t}$$

$$\frac{dR}{dt} = \lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t} = \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t}$$

Our system of equations become a system of **differential equations**

$$\begin{cases} \dfrac{dS}{dt} = -\beta \dfrac{I(t)}{N} S(t) \\ \dfrac{dI}{dt} = \beta \dfrac{I(t)}{N} S(t) - \gamma I(t) \\ \dfrac{dR}{dt} = \gamma I(t) \end{cases} \tag{3}$$

1.3 Condition for an epidemic

An epidemic occurs if the number of infected individuals increases initially, that is if

$$\beta \frac{I(0)}{N} S(0) - \gamma I(0) > 0,$$

and

$$S(0) > \frac{\gamma}{\beta} N.$$

The disease dies out if the number of infected individuals decreases initially that is if

$$\beta \frac{I(0)}{N} S(0) - \gamma I(0) < 0,$$

and

$$S(0) < \frac{\gamma}{\beta} N.$$

We call $\dfrac{\beta}{\gamma}$ the basic reproductive number $\mathcal{R}_0$. This is the number of new infectives caused by one single infective introduced into a population that is entirely susceptible during the time span of the infection.

Exercice 3 : We go back to our example of the rubella. Do we have an epidemic? Compute $\mathcal{R}_0$.

1.4 Equilibrium points

A point (S_{eq}, I_{eq}, R_{eq}) (where S_{eq}, I_{eq} and R_{eq} are constant) is an equilibrium or steady state for the system of equations (3) if S_{eq}, I_{eq} and R_{eq} verify

$$\begin{cases} \dfrac{dS}{dt} = 0 = -\beta \dfrac{I_{eq}}{N} S_{eq} \\ \dfrac{dI}{dt} = 0 = \beta \dfrac{I_{eq}}{N} S_{eq} - \gamma I_{eq} \\ \dfrac{dR}{dt} = 0 = \gamma I_{eq} \end{cases} \tag{4}$$

As t grows larger and larger, the solution to the epidemic model (S, I, R) eventually settles down to a steady state.

Exercice 3 : Find I_{eq}. What does it mean for the epidemic?

2 The endemic model

We add births and deaths to our system. We assume that births and deaths occurs at the same rate so that the size of the population as a whole does not change with time. We denote by μ the common constant birth and death rate. $\mu = \dfrac{1}{L}$, where L is the life expectancy in days.

$$\begin{cases} \dfrac{dS}{dt} = \mu N - \beta \dfrac{I(t)}{N} S(t) - \mu S(t) \\ \dfrac{dI}{dt} = \beta \dfrac{I(t)}{N} S(t) - \gamma I(t) - \mu I(t) \\ \dfrac{dR}{dt} = \gamma I(t) - \mu R(t) \end{cases} \tag{5}$$

Let $s(t) = \dfrac{S(t)}{N}, i(t) = \dfrac{I(t)}{N}$ and $r(t) = \dfrac{R(t)}{N}$ be respectively the fraction of the population that is susceptible, the fraction of the population that is infective and the fraction of the population that is removed. We divided the previous equations by N and get

$$\begin{cases} \dfrac{ds}{dt} = \mu - \beta\, i(t)s(t) - \mu\, s(t) \\ \dfrac{di}{dt} = \beta\, i(t)s(t) - \gamma\, i(t) \\ \dfrac{dr}{dt} = \gamma\, i(t) - \mu\, r(t) \end{cases} \tag{6}$$

Homework problem :

(a) Use the identity $N = S + R + I$ to prove $s + i + r = 1$.

(b) We suppose that the life expectancy is $L = 75$ years. Find μ. (Be careful with the units).

(c) We consider the same parameters as before. Complete the table below with the help of a calculator.

t	0	1	2	3	4
s					
i					
r					

(d) Does an epidemic occur in this case?

Bonus questions : We go back to the general case (6).

(a) Give a condition so that an epidemic occurs.

(b) Which equations does an equilibrium point (S_{eq}, I_{eq}, R_{eq}) verify in this model? Find all the equilibrium points possible. What does it mean for the epidemic.

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Mathematical modeling of an epidemic
Numerical Simulations

At time $t = 0$, our population is composed of $100,000$ healthy individuals and 1000 infected individuals.

1 The general epidemic model

Problem 1 : Differences in parameters and outcome of the epidemic

For each case, run the program for the following parameters β and τ

- A rubella epidemic : $\beta = 0.62$ and $\tau = 11$
- A measles epidemic : $\beta = 1.88$ and $\tau = 8$
- An influenza epidemic : $\beta = 0.47$ and $\tau = 3$

(a) Which one of these three diseases is highly contagious, which is moderately contagious and which is least contagious?

(b) Which one allows a larger portion of the population to escape infection? How many escape in each case?

Problem 2 : Does an epidemic occur or not?

(a) Find one set of parameters such that an epidemic occurs. Verify that the condition that we found last time is verified in this case.

(b) Find one set of parameters such that an epidemic does not occur. Verify that the condition that we found last time is not verified in this case.

2 The endemic model

We suppose that the life expectancy is 70 years.

Problem 3 : Modifying the code to add births and deaths in the population

Build a program to compute the evolution with time of the proportion of susceptibles, infected and removed in the population when you add births and deaths to the model. Plot these three functions on a common figure.

Problem 4 : Outcome of the epidemic

(a) Find two sets of parameters β and γ that lead to two different outcomes for the disease.

(b) How many people escape infection in each case?

(c) Verify that these two outcomes correspond to the two equilibrium points that you found for this system of differential equations.

Problem 5 : Does an epidemic occur or not?

(a) Find a condition on the initial value of s such that an epidemic occurs.

(b) Find one set of parameters such that an epidemic occurs. Verify that the previous condition is verified in this case.

(c) Find one set of parameters such that an epidemic does not occur. Verify that the previous condition is not verified in this case.

3 Introducing vaccination

We consider that a proportion p of the new born population are being vaccinated. We still assume that the population is constant.

Problem 6 :

(a) Write the system of differential equations that s, i and r verify with this new assumption.

(b) Build a program to compute the evolution with time of the proportion of susceptibles, infected and removed in the population in this case. Plot these three functions on a common figure.

Problem 7 : Does an epidemic occur or not?

(a) Find a condition on the initial value of s such that an epidemic does not occur.

(b) Find the initial value of s as a function of p in a population where no one is infected and you have a constant vaccination at birth.

(c) Find a condition on p such that an epidemic does not occur.

(d) Find a value of p such that an epidemic of rubella occurs. How many people escape infection?

(d) Find a value of p such that an epidemic of rubella does not occur. How many people escape infection?

(d) How many people in a population must be vaccinated in order to prevent the spread of an epidemic of rubella, measles and influenza?

Problem 8 : Outcome of the epidemic

We consider in this problem that $p = 0.9$.

(a) Find two sets of parameters β and γ that lead to two different outcomes for the disease.

(b) How many people escape infection in each case?

(c) Verify that these two outcomes correspond to the two equilibrium points of the system. Give an expression for these two equilibrium points.