Propositional Logic

Logic is the formal study of deductive reasoning. When mathematicians study logic, they are interested in understanding the steps of reasoning which form a valid proof, in the sense of proving mathematical theorems. Today we will explore the basic language which logicians have invented for describing the process of proving mathematical theorems. We start with propositional logic, which looks at the relationships between sentences. Let's start with some examples:

Thanksgiving Dessert Menu
Axioms:

- If we eat chocolate cake then we will eat ice cream.

- If we eat pumpkin pie, then we will eat whipped cream.

- We will not eat BOTH ice cream and whipped cream.

Now, Thursday is Thanksgiving.

1. Given the premise that *We will eat pumpkin pie*, prove that *We will not eat chocolate cake*.

2. Given the premise that *We will not eat chocolate cake*, can you prove that *We will not eat ice cream*? Why not?

Symbols and Truth Tables

Let's make symbols and a truth table for the axiom *If we eat chocolate cake then we will eat ice cream.* When we make a truth table for a sentence, we evaluate the truth of the sentence for any set of truth values for its constituent parts.

The symbol "C" stands for *We will eat chocolate cake.*

The symbol "I" stands for *We will eat ice cream.*

C	I	$C \to I$
T	T	
T	F	
F	T	
F	F	

Let's think of another example. Let's make a truth table for the sentence *If you are a bank robber, then you go to jail.*

R	J	$R \to J$	$\neg J \to \neg R$
T	T		
T	F		
F	T		
F	F		

If you are not a bank robber, do you go to jail? Can you think of someone who is not a bank robber who does go to jail?

What about *If you do not go to jail, then you are not a bank robber.* Does this follow from $R \to J$? Let's make a truth table for this statement. Do you notice anything similar between these two truth tables? The statements $R \to J$ and $\neg J \to \neg R$ are logically equivalent. One is true if and only if the other is true.

More Symbols and Truth Tables

Let's add some more logical operators into our language so that we will be able to say more.

Use the symbol M for *Math is cool,* S for *Science is cool* and R for *Reading is cool.*

The AND operator is M **and** S. It is true when **both** M and S are true. The OR operator is M **or** S. In logic, M **or** S is true if **at least one** of M or S is true. The NOT operator is $\neg M$. $\neg M$ is true if M is false and false if M is true.

Let's make truth tables for each of these operators.

M	S	M **and** S
T	T	
T	F	
F	T	
F	F	

M	S	M **or** S
T	T	T
T	F	
F	T	
F	F	

M	$\neg M$
T	
F	

Now we will practice writing sentences using the symbols of our logical language. Use **M, S,** and **R** as your sentence symbols.

1. Write the sentence *Math is cool and science is cool and reading is not cool* in symbols.

2. Write the sentence *If reading is cool, then (math is cool and science is not cool)* in symbols.

3. Write the sentence *Reading is cool and (science is cool or math is cool)* in symbols.

4. Write the sentence *If science is not cool, then (reading is not cool or math is cool)* in symbols.

5. Now, go back to our original axioms and write them using symbols.

- If we eat chocolate cake then we will eat ice cream.

- If we eat pumpkin pie, then we will eat whipped cream.

- We will not eat BOTH ice cream and whipped cream.

Let's make a truth table for a longer sentence, $\neg(I \text{ and } W)$. Which axiom of ours is this?

We will make the truth table in steps, one for the parentheses and then one for the $\neg$ sign.

I	W	$(I$ and $W)$	$\neg(I$ and $W)$
T	T		
T	F		
F	T		
F	F		

Here's one for the sentence $\neg(I)$ or $\neg(W)$.

I	W	$\neg(I)$	$\neg(W)$	$\neg(I)$ or $\neg(W)$
T	T			
T	F			
F	T			
F	F			

If you filled in the charts correctly, you will notice that the two sentences have the same truth table. This means that these two sentences are **logically equivalent**. Can you see why the sentence *We will not have BOTH ice cream and chocolate cake* is logically equivalent to *We will not have ice cream OR we will not have chocolate cake*?

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Exercises

6. Write your own truth table for ¬(*I* **or** *W*). What does this sentence mean in words? Extra credit: Can you find another sentence which is logically equivalent to this sentence? (Hint: Use the **and** operator)

I	W		¬(*I* **or** *W*)
T	T		
T	F		
F	T		
F	F		

I	W	¬(*I*)	¬(*W*)	
T	T			
T	F			
F	T			
F	F			

So ¬(*I* **or** *W*) is logically equivalent to _____.

7. Write your own truth table for *A* **and** (*B* **or** *C*). Here there are three different sentences, so we will need eight rows of our truth table.

A	B	C	(*B* **or** *C*)	
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

8. Write your own truth table for $(A \to B) \to A$.

A	B		
T	T		
T	F		
F	T		
F	F		

When is this sentence true?

A sentence which is true no matter what the truth values of its constituent sentences is called a tautology. Which of the examples which we just did is a tautology?

9. Show that the following sentence is a tautology: $(A \text{ and } B) \to A$.

A	B	A and B	
T	T		
T	F		
F	T		
F	F		

Riddles

Riddle 1: Who Loves Tennis?

An advertisement for a tennis magazine says, *If I'm not playing tennis, I'm watching tennis. And if I'm not watching tennis, I'm reading about tennis.* Assume that the speaker can only do one activity at once. What is the speaker doing? Try writing out the sentences in symbols and see what you get:)

Why does the truth table only require 3 rows?

P	W	R	$\neg(P) \to W$	$\neg(W) \to R$	

Riddle 2: Truthteller or Liar

There are two tribes of trolls, the Glums and the Plogs. Glums are all truthtellers and Plogs are all liars. You meet two trolls one day, Kim and Coin. Kim says "We are from different clans." Coin says "Kim is a liar." Which tribe is Kim from and which tribe is Coin from? Here is a table:

K	C	Different Clans	Kim is a liar
G	G		
G	P		
P	G		
P	P		

Riddle 3: Desert Island

You are stuck on a desert island where people always tell the truth or always tell lies. You are at a fork in the road and you need to know the way back to the ship. There is a resident there but he has time only to reply to one yes or no question. What one question should you ask to figure out which way to go? (Make a table.)

Let P be the sentence: *The people are truth tellers.*

Let W be the sentence: *The right way is to the right.*

Here is a table:

P	W	P and W	$\neg(P)$ and $\neg(W)$	$(P$ and $W)$ or $(\neg(P)$ and $\neg(W))$
T	T			
T	F			
F	T			
F	F			

Why is this the right question?

Math Example

Two lines are parallel if they never intersect. The sum of the interior angles between A and B is the sum of the two angles we get from drawing a line which crosses A and B.

Axioms:

- If two lines are parallel then they are at constant distance from each other.

- If the sum of the interior angles between two lines is not 180° then they are not at constant distance from each other.

Write these sentences in symbols. Then, given the premise that *The sum of the interior angles between two lines is not* 180° then prove that *They two lines are not parallel*. Note that these statements are equivalent versions of Euclid's parallel line axiom, which was debated for several thousand years, until mathematicians realized that non-Euclidean geometries do not respect this axiom!