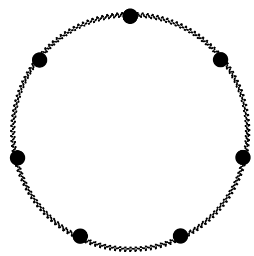
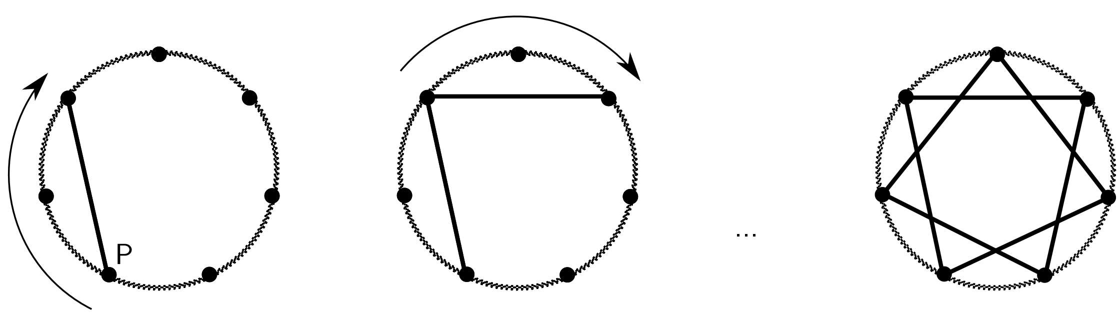
MATH CIRCLE ACTIVITY: STARS AND CYCLIC GROUPS

1. Drawing a regular star

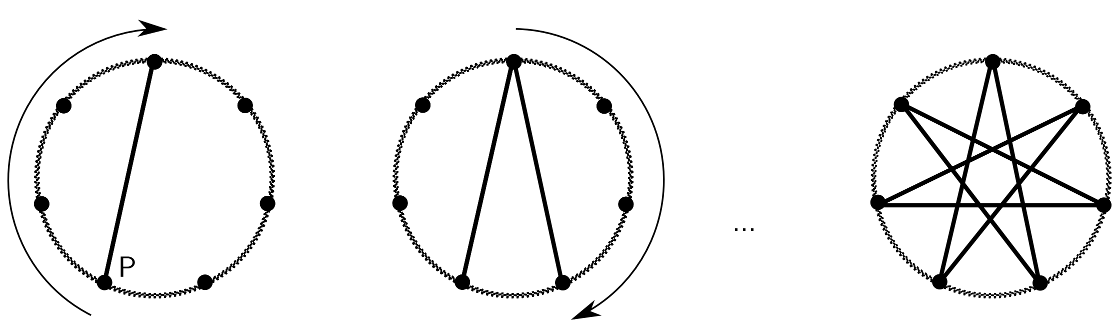
A **regular star** is a self-intersecting polygon which is symmetric around its center. Let's start with a concrete example. Take a circle, draw seven points equally spaced (for simplicity's sake, we will start with one point at the very top).



Start at any point P, count over two points clockwise, and connect this point to your starting point. From this new point, count over two points again, connect this third point to your second. Continue this pattern until you reach the start



This method has given us a seven-pointed star! Try this method again, except now count over three points at each step.



We get a different seven-pointed star! We will call this star 7 count 3 (7:3) and we will call the first star 7 count 2 (7:2). We can try a similar method for any whole number of n points and and any count k to create a polygon $(n : k)$.

Problem 1.1 (Generating a star)**. Given a number of points $\mathbf{n}$ on a circle and a clockwise count $\mathbf{k}$ to connect the points, will $(\mathbf{n} : \mathbf{k})$ always be a star?**

To make our question more specific, we will call n count k $(n : k)$ the regular polygon formed by the pattern above. We will call $(n : k)$ a **regular star** if $(n : k)$ intersects itself. We will also refer to $(n : k)$ as an $\mathbf{m}$**-star** if $(n : k)$ is an m-pointed regular star.

We will explore the problem by filling in the table on the following page. Here are the instructions to create $(n : k)$:

- Take n points and space them out on your circle (for simplicity, put one of your points at the top of your circle). It is okay if the spacing is not exactly even.
- Pick a starting point (for instance, the top point), count over k points clockwise and connect this point to your starting point.
- Count k points clockwise from your second point to your third and connect the second and third points.
- Once you are connected to your starting point again, stop drawing even if there are unconnected points left.

(Points:count)	**(n:k) drawing**	**# Points connected**	**Is it a star?**
$(\mathbf{7}:\mathbf{1})$		7	No
$(\mathbf{5}:\mathbf{2})$		5	Yes
$(\mathbf{6}:\mathbf{2})$			
$(\mathbf{8}:\mathbf{3})$			
$(\mathbf{10}:\mathbf{2})$			
$(\mathbf{10}:\mathbf{4})$			
$(\mathbf{99}:\mathbf{3})$	Too many points to draw!		

Problem 1.2 (Stars and Number Patterns)**.** In the table above, we saw some patterns that didn't create a star. We saw some patterns create smaller stars, and we saw two different patterns generate the same star. We can ask many questions about these patterns, such as:

- Can we tell if $(n : k)$ is a star?
- How many endpoints will $(n : k)$ have?
- Is there k where $(8 : k)$ will have 5 endpoints? What about 4 endpoints?
- Suppose we take a prime number p? Will $(p : k)$ always be a p-star?
- How many times will we wind around the circle when we draw $(n : k)$?
- Is there some n where $(n : k)$ will never be an n-star? If so, how many n's are there?
- How can we tell if any two different patterns give the same star? Try to find three different patterns which give the same star.
- How can we tell if $(n : k)$ and $(n : j)$ give the same star?
- How many stars can we generate with by changing k in $(n : k)$?
- How many n-stars can we generate with different values of k in $(n : k)$?
- What percentage of the time will $(n : 2)$ create an n-star for different n?

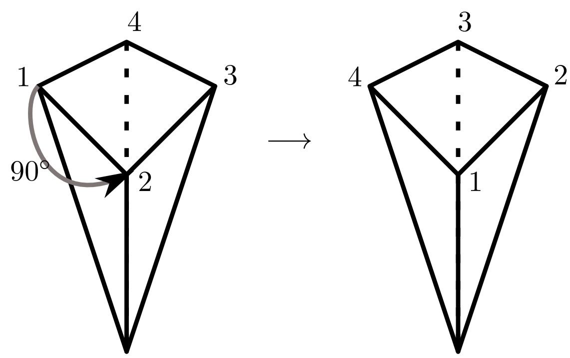
To investigate these questions on big numbers, it is too hard to draw everything out by hand. The good news is that there is that we can answer these questions by looking at how n and k are related. To develop this relationship, we will need to be familiar with divisors and with the greatest common divisor of two numbers.

2. **Cyclic Groups, generators, and subgroups**

A **group** is a list of actions on an object or a set which follow these rules:

(1) The list is predefined and never changes.
(2) Every action can be undone by another action.
(3) Every action is deterministic.
(4) Any two consecutive actions is also an action.

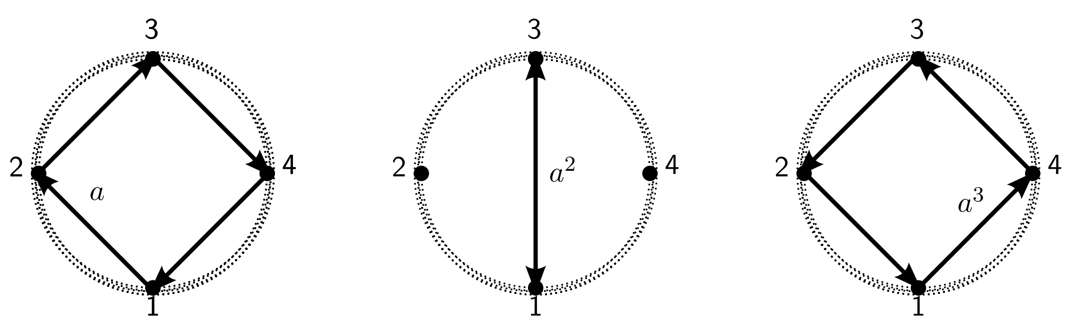
A **cyclic group** is a group where every nontrivial action can be generated by a single action or its reverse action. For example, consider a square dial stuck in a wall, and consider the clockwise rotations of this dial which preserve the orientation of the vertices and edges.



Notice that we have exactly 4 actions on this dial. What are they? Notice that every clockwise rotation can be undone by another clockwise rotation. Which one? Notice that we can always tell how the rotation will change our vertices. Notice that every two rotations is another rotation. And finally, notice that every rotation can be achieved by rotating clockwise $90°$ multiple times.

We can relate these actions on this dial to a set $C_4 = e, a, a^2, a^3$, where e describe no rotation and a is rotation by $90°$. Note that this means $a^4 = e$, since rotating all the way around creates the same position as no rotation.

Now C_4 doesn't only act on dials by rotation. Consider 4 points lying on a circle, which we will label clockwise p_0, p_1, p_2, and p_3. C_4 acts on these points by clockwise travel between points.



Notice these are the same paths that we created when we were drawing stars (except there are no 4-stars). Acting on n points with a^k in C_n is the same as taking a count k along the points. This shows that a cyclic group C_n can be interpreted many ways and applied to more than one situation!

Element of Cyclic Group	**Action on points**	**Order of element**	**Actions Generated by element**
a in C_9		9	$a, a^2, a^3, a^4,$ a^5, a^6, a^7, a^8, e
a^4 in C_9		9	$a, a^2, a^3, a^4,$ a^5, a^6, a^7, a^8, e
a^3 in C_9		3	a^3, a^6, e
a^{10} in C_{13}			
a^4 in C_{14}			
a^3 in C_6			
a^{11} in C_{121}	Too many points to draw!		

We can use a star's structure to uncover the structure of a cyclic group! Notice that some actions don't generate the entire group. These elements only generate part of the group known as a **subgroup**.

Consider the following exercises:

- How many generators does C_5 have?
- Find another n where C_n is generated by a^3.
- Does a^2 generate all of C_{10}? What points does a^2 connect if you start at p_0 and keep traveling by a^2? What actions are created by a^2?
- What are the subgroups of C_8?
- For C_n, how many points does repeated action by a^k connect?
- For C_n, how can we tell if a^k's action traces out a star?