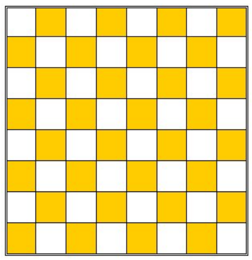
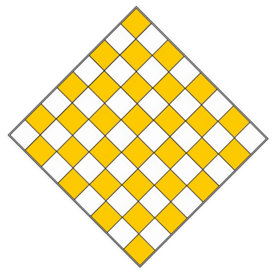
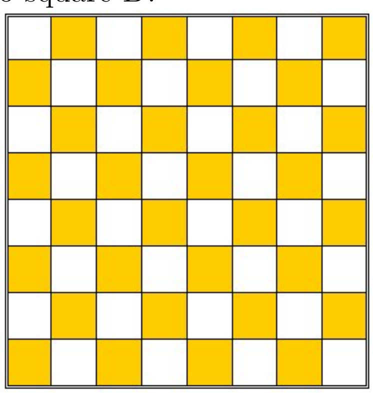
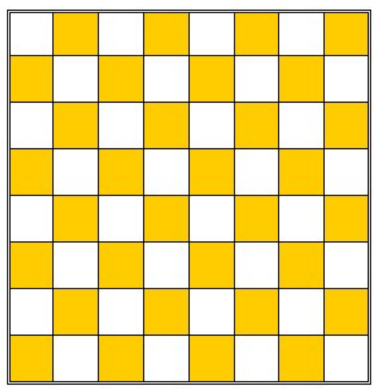
UCI Math Circle – Taxicab Geometry
The Chessboard Distance

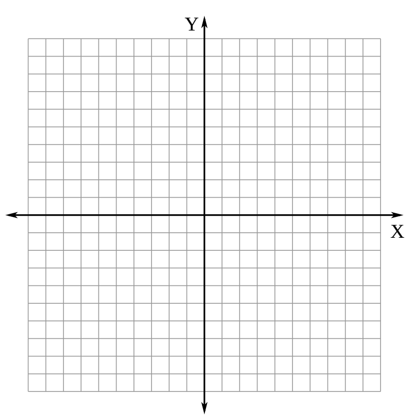
1. There are two chess pieces which would measure distances using taxicab geometry. Can you think which two pieces they are?

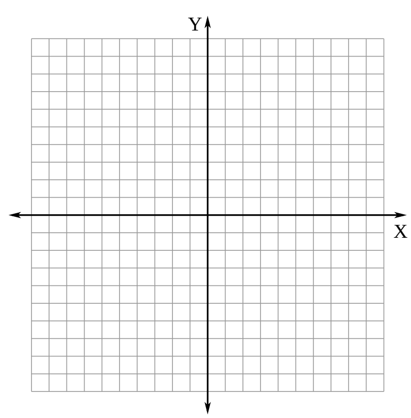




2. Now think of the king. How many moves would it take to move the king from square A to square B?

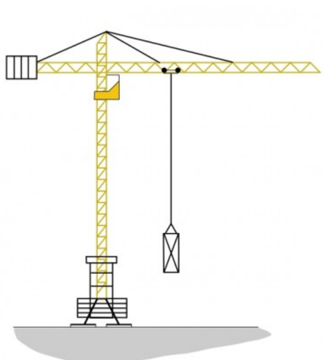



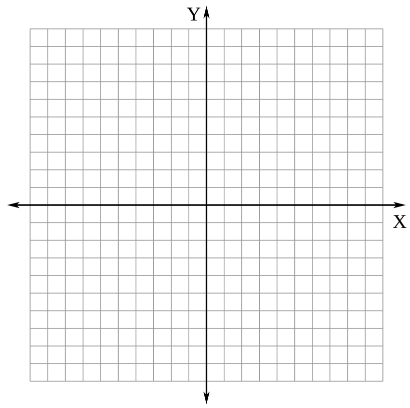

3. Let's move to our coordinate axes. How many moves would it take to move the king from point A to point B? We call this the chessboard distance.

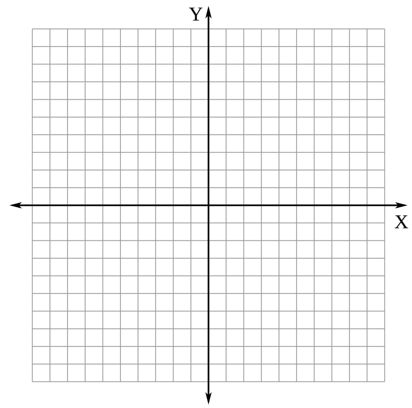




4. There is another way to think of the chessboard distance. Think of a crane which needs to travel from A to B. It can move along the horizontal and vertical axes at the same time. How many moves would it take a crane to get from A to B? This will just be the larger of the x-distance and the y-distance. Why?







Now let's formulate it mathematically. We have the points $A = (a, b)$ and $B = (c, d)$.

We already know that the taxicab distance is $d_{taxi}(A, B) = |a - c| + |b - d|$ and the Euclidean distance is $d_{Eucl}(A, B) = \sqrt{|a - c|^2 + |b - d|^2}$. How would we describe the chessboard distance mathematically?

$d_{chess}(A, B) =$

5. Let $A = (-1, 2)$ and $B = (2, 6)$. Find the taxicab distance, the chessboard distance, and the Euclidean distance between these two points.

$|2 - (-1)| =$ $\qquad\qquad$ $|6 - 2| =$

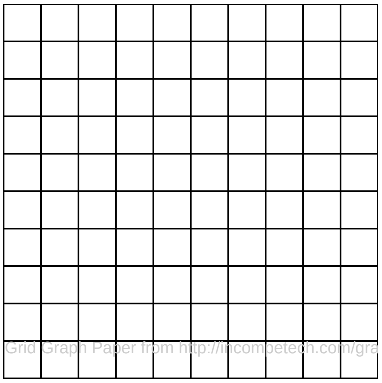
$d_{taxi}(A, B) =$

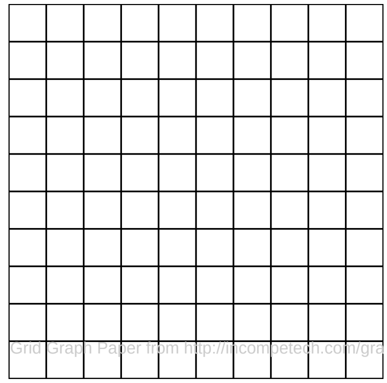
$d_{chess}(A, B) =$

$d_{Eucl}(A, B) =$

6. Let $A = (-3, -4)$ and $B = (3, 4)$ Find the taxicab distance, the chessboard distance, and the Euclidean distance between these two points.

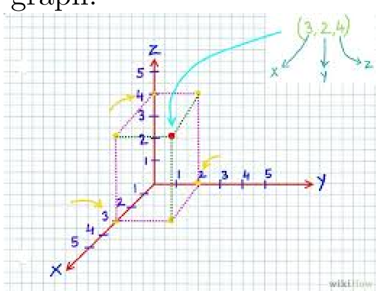
7. Draw a circle of radius two for the chessboard distance. Then draw the Euclidean circle and the taxicab circle within it. In the second coordinate chart, draw a chessboard circle of radius four, with a Euclidean and taxicab circle within it.

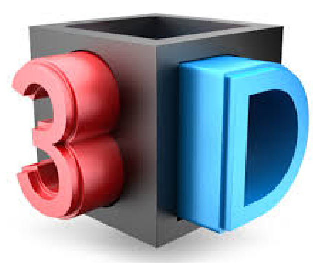




8. What is the value of π in the chessboard distance?

9. Now we are going to make the big jump from 2-D to 3-D. In 3-D, we use 3 coordinates, x, y, and z. For example, here is the point $A = (3, 2, 4)$. Add in the point $B = (3, 4, 4)$ to the graph.





We can use the same distance formulas which we used in 2-D for 3-D. Let's see. We have points $A = (a, b, c)$ and $B = (d, e, f)$.

Then the taxicab distance is $d_{taxi}(A, B) = |a - d| + |b - e| + |c - f|$.

The chessboard distance is $d_{chess}(A, B) = Maximum(|a - d|, |b - e|, |c - f|)$

The Euclidean distance is $d_{Eucl}(A, B) = \sqrt{|a - d|^2 + |b - e|^2 + |c - f|^2}$

10. Let $A = (2, 1, 3)$ and $B = (4, 5, 7)$. Calculate the distance between A and B in all three distances.

$d_{taxi}(A, B) = |4 - 2| + |5 - 1| + |7 - 3| =$

$d_{chess}(A, B) = Maximum(|4 - 2|, |5 - 1|, |7 - 3|) =$

$d_{Eucl}(A, B) = \sqrt{|4 - 2|^2 + |5 - 1|^2 + |7 - 3|^2} =$

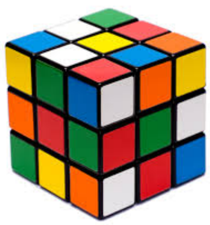
11. Now let $A = (-1, 6, 1)$ and $B = (7, -2, 5)$. Calculate the distance between A and B in all three distances.

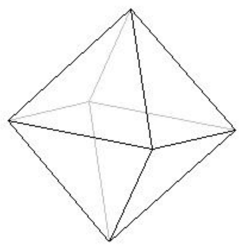
$d_{taxi}(A, B) =$

$d_{chess}(A, B) =$

$d_{Eucl}(A, B) =$

12. Now we will generalize the idea of a circle in 3-D. A circle in 3 dimensions is called a sphere. The sphere of radius 2 is the set of all points that are a distance 2 from the center in 3-D. Here are spheres for each of our three distances. Match each sphere with the proper distance. How did you know which one goes with which?

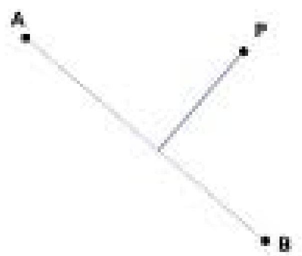




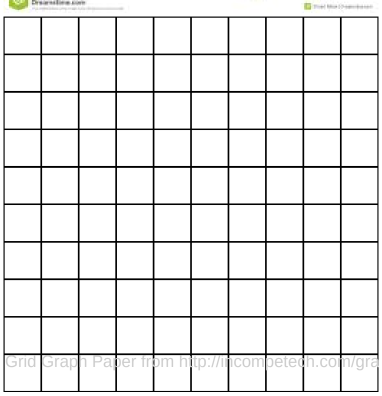
UCI Math Circle – Taxicab Geometry Exercises

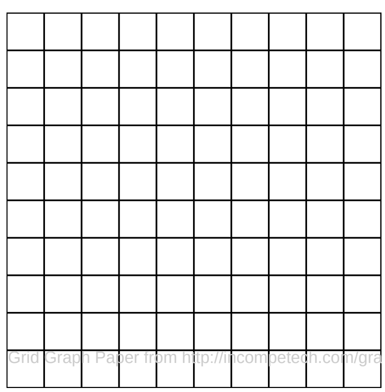
Here are several more exercises on taxicab geometry.

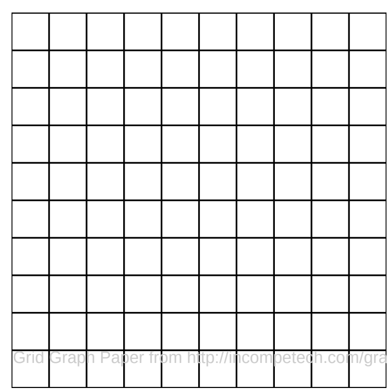
1. In Euclidean geometry, the distance between a point and a line is the length of the perpendicular line connecting it to the plane.



This is not true in taxicab geometry. In the following 3 pictures, the diagonal line is Broadway Street. You are at the point P. Your friend Samantha wants to meet somewhere on Broadway street. What is the shortest way to get to Broadway from where you are standing?

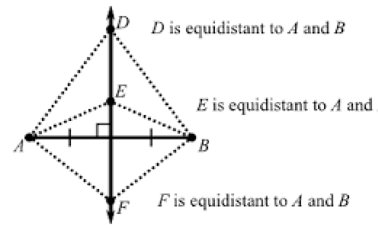






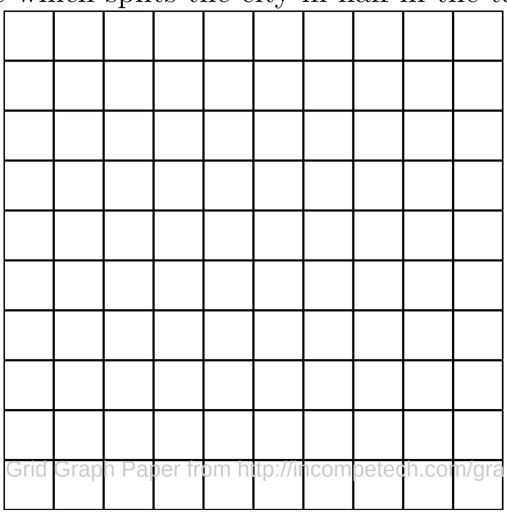
2. Do you see a pattern as to when the shortest path is a horizontal line and when the shortest path is a vertical line? (Hint: Think of the slope of line which is Broadway Street.)

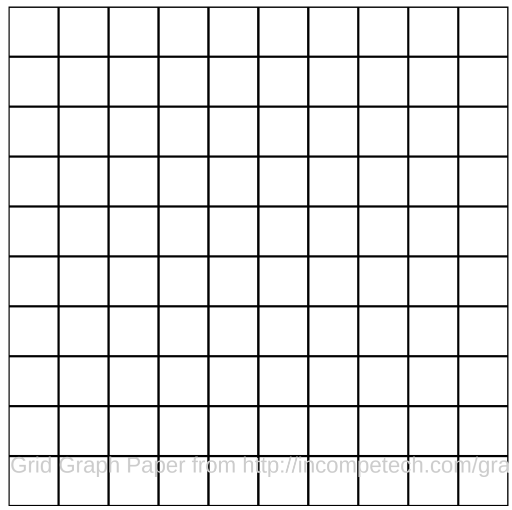
3. You are a fire chief and you have two fire stations in your city, Fire Station A and Fire Station B. You need to split your city in half so that each fire station is responsible for the area that is closest to him.

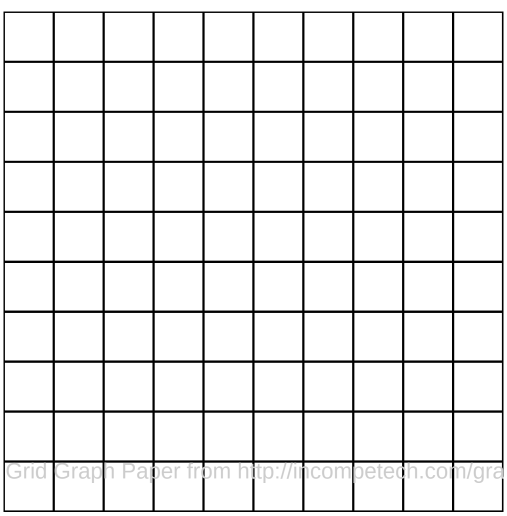


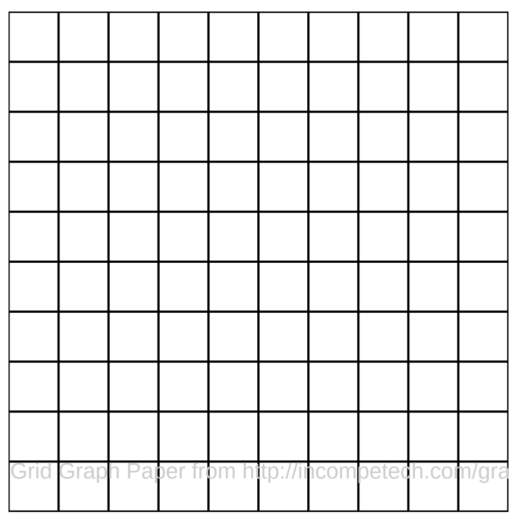
For the Euclidean distance, we would have the following line.

But the fire chief need to use the taxicab distance. In the following four pictures draw a line which splits the city in half in the taxicab distance.

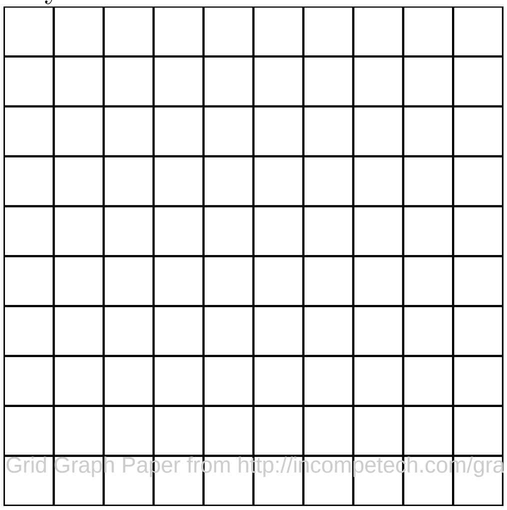


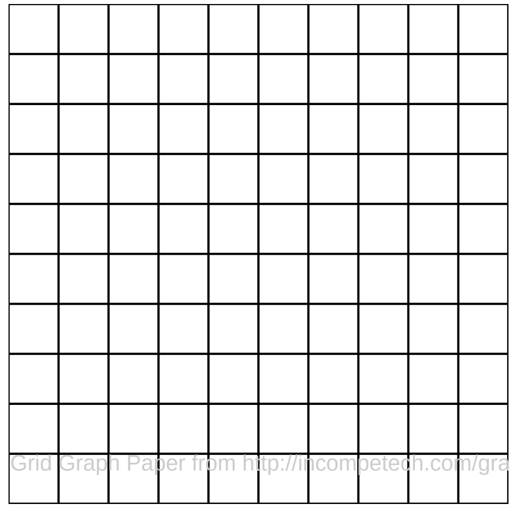






4. Now for a challenge problem we will have 3 fire stations and you should try to divide the city in 3 zones.

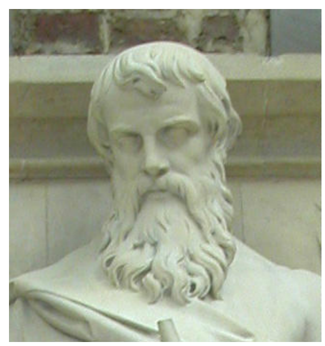




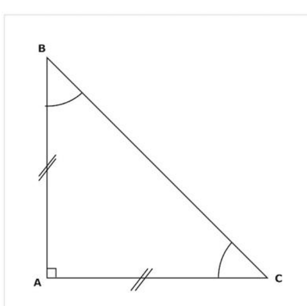
5. **What is wrong with taxicab geometry?**

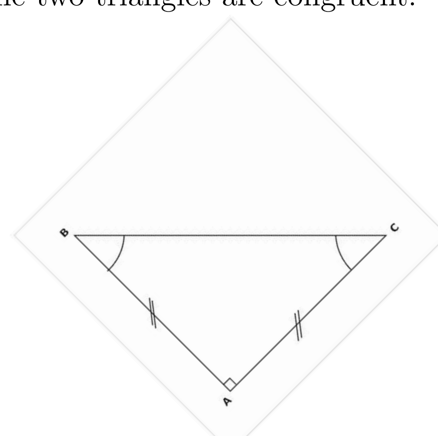
Althoug taxicab geometry is really cool and is useful for taxi drivers, mathematicians don't really use it that much. Euclidean geometry is much more useful for studying the world. Why? Let's see an example.

Taxi vs. Euclid

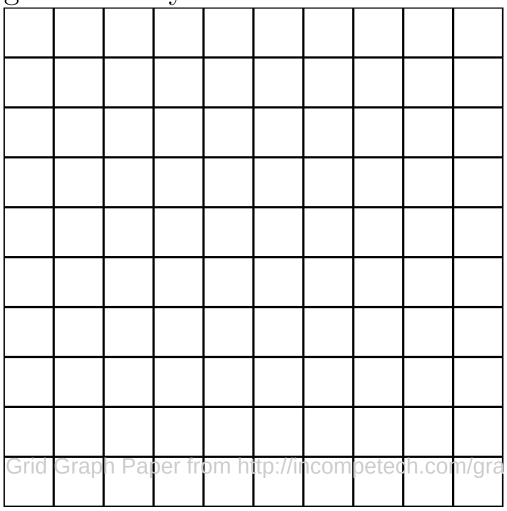


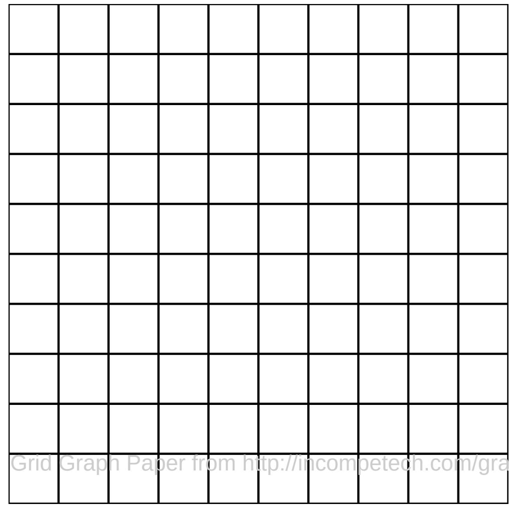
Look at the two Euclidean triangles below. The two triangles have SAS congruence. We know from Euclidean geometry that the two triangles are congruent.





Now lets look at two SAS conguent triangles in taxicab geometry. Are the two triangles congruent? Why not?





In taxicab geometry, shapes change when we rotate them. This is because in taxicab geometry you can only travel along the x and y axes. This means that the rules of the geometry change depending which way you are facing. The x and y directions are more important than any other directions.

In the real world, there is no way to pick two direction as more important than any other direction. We want our physical laws to be independent of the coordinate system which we use. In the middle of space, there would be no way to pick out which way is the x and which way is the y direction. But it still is very useful for cabbies!