HS Math Circle

February 1st, 2016

Notations:

(i) denote the greatest common divisor of a and b by (a, b).
(ii) write $a \equiv b \mod m$ if the remainder of a and b divided by m are equal.

Problem 1. (Level: ★★)

(a) Find all pairs of integers (a, b) which satisfy

$$ab - 2a - 4b = 5.$$

Hint: there are several ways to do this problem. One way is to change the equation to

$$(\cdots) \times (\cdots) = \text{constant}.$$

(b) Find all positive pairs of integers (a, b) which satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{6}.$$

(c) Find all positive pairs of integers (a, b, c) which satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Hint: this is difficult. First, notice that without loss of generality we can assume that $a \leq b \leq c$. Then a cannot be very large. Why? :)

(d) Find all positive integer n such that

$$\sqrt{n^2 + 99}$$

is an integer.

Hint: letting $\sqrt{n^2 + 99} = m$ and ...what?

(e) Find all positive integer n such that

$$\sqrt{n^2 + n + 14}$$

is an integer.

Hint: the same as (d), but then you need to c...

Problem 2. (Level: ★★★)

(a) Compute

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}.$$

(b) Similarly, compute

$$y = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}.$$

(c) Conversely, can you express, for example $\sqrt{14}$, in the above ways?

(d) Let us write

$$\frac{a_1}{b_1} = 1, \quad \frac{a_2}{b_2} = 1 + \frac{1}{1}, \quad \frac{a_3}{b_3} = 1 + \cfrac{1}{1 + \cfrac{1}{1}}, \quad \frac{a_4}{b_4} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}, \quad \frac{a_5}{b_5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}},$$

and so on, where a_n and b_n are positive integers with $(a_n, b_n) = 1$. Compute a_n and b_n for $n = 1, 2, 3, 4, 5$. Do you see any patterns? If so, can you prove it?

(e) Similarly, let us write

$$\frac{p_1}{q_1} = 1, \quad \frac{p_2}{q_2} = 1 + \frac{1}{2}, \quad \frac{p_3}{q_3} = 1 + \cfrac{1}{2 + \cfrac{1}{1}}, \quad \frac{p_4}{q_4} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}, \quad \frac{p_5}{q_5} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}},$$

and so on, where p_n and q_n are positive integers with $(p_n, q_n) = 1$.

(i) Compute p_n and q_n for $n = 1, 2, 3, 4, 5$. Do you see any patterns? If so, can you prove it?

(ii) Consider

$$x^2 - 2y^2 = \pm 1.$$

Find nonnegative integer solutions (there are infinitely many). Do you see any patterns? Can you prove it?

Problem 3. (Level: ★★★★)

(a) Compute

$$5 \cdot i \mod 8$$

for $i = 1, 2, \cdots, 7$.

(b) Compute

$$8 \cdot i \mod 6$$

for $i = 1, 2, \cdots, 5$.

(c) What can we say from (a) and (b)? Can you formulate any theorem? Can you prove it?

(d) Let us assume that $(a, b) = 1$. Compute

$$\left[\frac{a}{b}\right] + \left[\frac{2a}{b}\right] + \cdots + \left[\frac{(b-1)a}{b}\right].$$

Hint: this problem is difficult. First, compute

$$\frac{a}{b} + \frac{2a}{b} + \cdots + \frac{(b-1)a}{b}.$$

(e) Is it possible to compute (d) in a different way? :)

(f) Let p be a prime number, and assume that $(a, p) = 1$. Show that

$$a^{p-1} \equiv 1 \mod p.$$