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Arithmetic spectral transitions: a competition between hyperbolicity and the arithmetics of small denominators

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1. Introduction

Unlike random, one-dimensional quasiperiodic operators feature spectral transitions with changes of parameters. The transitions between absolutely continuous and singular spectra are governed by vanishing/non-vanishing of the Lyapunov exponent. In the regime of positive Lyapunov exponents there are also

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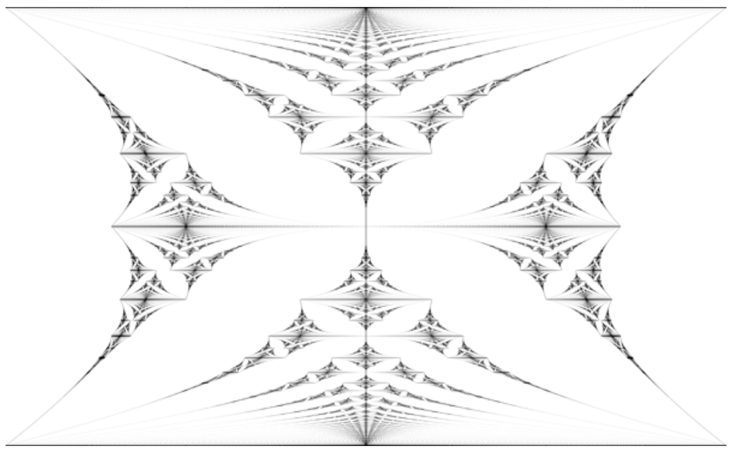
more delicate transitions: between localization (point spectrum with exponentially decaying eigenfunctions) and singular continuous spectrum, and dimensional/quantum dynamics transitions within the regime of singular continuous spectrum, governed by the arithmetics. Delicate dependence of spectral properties on the arithmetics is perhaps the most mathematically fascinating feature of quasiperiodic operators, made particularly prominent by Douglas Hofstadter's famous plot of spectra of the almost Mathieu operators, the Hofstadter's butterfly [22], see Fig. 1, demonstrating their self-similarity governed by the continued fraction expansion of the magetic flux. This self-similarity is even more remark-



able because it appears even in various experimental and quantum computing contexts, see e.g. Fig. 2

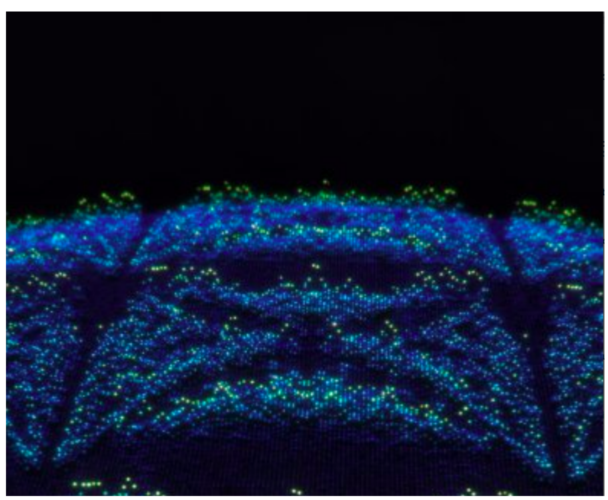


FIGURE 1.0.1. Photon spectrum simulated using a chain of 9 super-conducting quantum qubits [41]

Remarkably, such self-similarity of both spectra and eigenfunctions were predicted a dozen years before Hofstadter in the work of Mark Azbel [10], which, according to Hofstadter, was way ahead of its time. The self-similar behavior of eigenfunctions reflects the self-similar nature of resonances that are in competition to hyperbolicity provided by the Lyapunov growth. This competition also leads to the sharp transition between pure point (hyperbolicity wins) and singular continuous (resonances win) spectra in the positive Lyapunove exponent regime.

In the first three lectures we will outline a method to prove 1D Anderson localization in the regime of positive Lyapunov exponents that has allowed to solve the sharp arithmetic spectral transition (from absolutely continuous to singular continuous to pure point spectrum) problem for the almost Mathieu operator, in coupling, frequency and phase, and to describe the self-similar structure of localized eigenfunctions. The method is an adaption of [27, 28], but has its roots in [34] and even [33], with an important development in [4]. The last lecture will be devoted to the opposite goal: a method to prove certain delocalization within the regime of singular continuous spectrum (after [32]), that allowed to obtain a sharp arithmetic spectral transition result for the entire class of analytic quasiperiodic potentials.

2. The basics

2.1. Spectral measure of a selfadjoint operator Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$. The time evolution of a wave function is described in the Schrödinger picture of quantum mechanics by

$$i\frac{\partial \psi}{\partial t} = H\psi.$$

The solution with initial condition $\psi(0) = \psi_0$ is given by

$$\psi(t) = e^{-itH}\psi_0.$$

By the spectral theorem, for any $\psi_0 \in \mathcal{H}$, there is a unique spectral measure μ_{ψ_0} such that

$$(e^{-itH}\psi_0, \psi_0) = \int_{\mathbb{R}} e^{-it\lambda} d\mu_{\psi_0}(\lambda). \tag{2.1.1}$$

2.2. Spectral decompositions Let $\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{ac}$, where

$$\mathcal{H}_\gamma = \{\phi \in \mathcal{H} : \mu_\phi \text{ is } \gamma\}$$

and $\gamma \in \{pp, sc, ac\}$.

Here pp (sc, ac) is an abbreviation for pure point (singular continuous, absolutely continuous).

Operator H preserves each $\mathcal{H}_\gamma$, where $\gamma \in \{pp, sc, ac\}$. We then define: $\sigma_\gamma(H) = \sigma(H|_{\mathcal{H}_\gamma})$, $\gamma \in \{pp, sc, ac\}$. The set $\sigma_{pp}(H)$ admits a direct characterization as the closure of the set of all eigenvalues

$$\sigma_{pp}(H) = \overline{\sigma_p(H)},$$

where

$$\sigma_p(H) = \{\lambda : \text{ there exists a nonzero vector } \psi \in \mathcal{H} \text{ such that } H\psi = \lambda\psi\}.$$

2.3. Ergodic operators We are going to study Schrödinger operators with potentials related to dynamical systems. Let $H = \Delta + V$ be defined by

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{2.3.1}$$

on a Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$. Here $V : \mathbb{Z} \to \mathbb{R}$ is the potential. Let (Ω, P) be a probability space. A measure-preserving bijection $T : \Omega \to \Omega$ is called ergodic, if any T-invariant measurable set $A \subset \Omega$ has either $P(A) = 1$ or $P(A) = 0$. By a dynamically defined potential we understand a family $V_\omega(n) = \nu(T^n\omega), \omega \in \Omega$, where $\nu : \Omega \to \mathbb{R}$ is a measurable function. The corresponding family of operators $H_\omega = \Delta + V_\omega$ is called an ergodic family. More precisely,

$$(H_\omega u)(n) = u(n+1) + u(n-1) + \nu(T^n\omega)u(n). \tag{2.3.2}$$

Theorem 2.3.3 (Pastur [40]; Kunz-Souillard [36])**.** *There exists a full measure set Ω_0 and $\sum$, $\sum_{pp}$, $\sum_{sc}$, $\sum_{ac}$ such that for all $\omega \in \Omega_0$, we have $\sigma(H_\omega) = \sum$, and $\sigma_\gamma(H_\omega) = \sum_\gamma$, $\gamma = pp, sc, ac$.*

Theorem 2.3.4. *[Avron-Simon [9],Last-Simon [38]]If T is minimal, then $\sigma(H_\omega) = \sum$, and $\sigma_{ac}(H_\omega) = \sum_{ac}$ for all $\omega \in \Omega$.*

Theorem 2.3.4does not hold for $\sigma_\gamma(H_\omega)$ with $\gamma \in \{sc, pp\}$ [31], but it is an interesting and difficult open problem whether it holds for $\sigma_{sing}(H_\omega)$.

2.4. Schnol's theorem Let $H = \Delta + V$ be a Schrödinger operator on $\ell^2(\mathbb{Z})$. We say u is a generalized eigenfunction and E is the corresponding generalized eigenvalue if $Hu = Eu$ and $|u(n)| \leqslant C(1 + |n|)^{\frac{1}{2}+\epsilon}$ for some $C, \epsilon > 0$.

Theorem 2.4.1 (Schnol's theorem)**.** *Let S be the set of all generalized eigenvalues. For any $\psi \in \ell^2(\mathbb{Z})$, the spectral measure μ_ψ gives full weight to S and $\sigma(\Delta + V) = \overline{S}$.*

Here we modify the definition a little bit to avoid unnecessary notations. We will say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \leqslant \hat{C}(1 + |k|). \tag{2.4.2}$$

In the following, we usually normalize $\phi(k)$ so that

$$\phi^2(0) + \phi^2(-1) = 1. \tag{2.4.3}$$

2.5. Anderson Localization We say a self-adjoint operator H on $\ell^2(\mathbb{Z}^d)$ satisfies Anderson localization if H only has pure point spectrum and all the eigenfunctions decay exponentially. By Schnol's theorem, in order to show the Anderson localization of H, it suffices to prove that all polynomially bounded eigensolutions are exponentially decaying.

This can be done by establishing exponential off-diagonal decay of Green's functions. Block-resolvent expansion, a form of which we are about to see, is the backbone of Fröhlich-Spencer's multi-scale analysis, allowing to pass from smaller to larger scales and from local to global decay. The form we present, first developed for the almost Mathieu operator [33, 34], includes an important modification of multi-scale analysis type arguments, in simultaneously considering shifted boxes. This is the central ingredient in nonperturbative proofs for deterministic potentials [11].

For an interval I, let $G_I = (R_I(H_x - I)R_I)^{-1}$ if well defined (G_I is called the Green's function).

Definition 2.5.1. Fix $\tau > 0$, $0 < \delta < 1/2$. A point $y \in \mathbb{Z}$ will be called (τ, k, δ) regular if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1,x_2]}(y, x_i)| \leqslant e^{-\tau|y-x_i|} \text{ and } |y - x_i| \geqslant \delta k \text{ for } i = 1, 2.$$

This definition can be easily made multi-dimensional, with obvious modifications. The following argument is also multi-dimensional but we present a 1D version for simplicity..

First note that for $H\phi = E\phi$, we have $\phi = G_I\Gamma_I\phi$ where Γ_I is the decoupling operator at the boundary of I. In one dimensional case this reads

$$\phi(x) = -G_{[x_1,x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\phi(x_2 + 1), \tag{2.5.2}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

Theorem 2.5.3. *Let* $h(k) \to \infty$ *as* $k \to \infty$. *Suppose* $H\phi = E\phi$ *and* ϕ *satisfies* (2.4.2). *Suppose for any large* $k \in \mathbb{Z}$, k *is* (τ, y, δ) *regular for some* $h(k) \leqslant y \leqslant k$. *Then* H *satisfies Anderson localization. Moreover for any eigenfunction,* $\limsup_n \frac{\ln|\phi(n)|}{n} \leqslant -\tau$.

Proof. : Under the assumptions, there is some $\hat{k} \geqslant \delta \min_{y\in[\sqrt{k},2k]} h(y)$ such that for any $y \in [\sqrt{k}, 2k]$, there exists an interval $I(y) = [x_1, x_2] \subset [-4k, 4k]$ with $y \in I(y)$ such that

$$\operatorname{dist}(y, \partial I(y)) \geqslant \hat{k} \tag{2.5.4}$$

and

$$|G_{I(y)}(y, x_i)| \leqslant e^{-\tau|y-x_i|},\ i = 1, 2. \tag{2.5.5}$$

Denote by $\partial I(y)$ the boundary of the interval $I(y)$. For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 < 2k$ or $x_1 - 1 > \sqrt{k}$, we can expand $\phi(x_2 + 1)$ or $\phi(x_1 - 1)$ as (2.5.2). We can continue this process until we arrive to z such that $z + 1 \geqslant 2k$ or $z - 1 \leqslant \sqrt{k}$, or the iterating number reaches $[\frac{2k}{\hat{k}}]$, where $[t]$ denotes the greatest integer less than or equal to t.

By (2.5.2),

$$|\phi(k)| = |\sum_{s; z_{i+1}\in\partial I(z_i')} G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\phi(z_{s+1}')|, \tag{2.5.6}$$

where in each term of the summation we have $\sqrt{k}+1 < z_i < 2k-1$, $i = 1,\cdots,s$, and either $z_{s+1} \notin [\sqrt{k}+2, 2k-2]$, $s+1 < [\frac{2k}{\hat{k}}]$; or $s+1 = [\frac{2k}{\hat{k}}]$.

If $z_{s+1} \notin [\sqrt{k}+2, 2k-2]$, $s+1 < [\frac{2k}{\hat{k}}]$, by (2.5.5) and noting that $|\phi(z_{s+1}')| \leqslant (1+|z_{s+1}'|)^C \leqslant k^C$, one has

$$\begin{aligned}
&|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\phi(z_{s+1}')| \\
&\qquad \leqslant e^{-\tau(|k-z_1|+\sum_{i=1}^{s}|z_i'-z_{i+1}|)}k^C \\
&\qquad \leqslant e^{-\tau(|k-z_{s+1}|-(s+1))}k^C \\
&\qquad \leqslant \max\{e^{-\tau(k-\sqrt{k}-4-\frac{2k}{\hat{k}})}k^C, e^{-\tau(2k-k-4-\frac{2k}{\hat{k}})}k^C\}.
\end{aligned} \tag{2.5.7}$$

If $s+1 = [\frac{2k}{\hat{k}}]$, using (2.5.4) and (2.5.5), we obtain

$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\phi(z_{s+1}')| \leqslant k^C e^{-\tau\hat{k}[\frac{2k}{\hat{k}}]}. \tag{2.5.8}$$

Finally, notice that the total number of terms in (2.5.6) is at most $2^{[\frac{2k}{\hat{k}}]}$. Combining with (2.5.7) and (2.5.8), since $k/\hat{k} = o(k)$, we obtain for any $\varepsilon > 0$,

$$|\phi(k)| \leqslant e^{-(\tau-\varepsilon)k}$$

for large enough k . For $k < 0$, the proof is similar. Thus one has

$$|\phi(k)| \leqslant e^{-(\tau-\varepsilon)|k|} \text{ if } |k| \text{ is large enough.} \tag{2.5.9}$$

□

Therefore we only need to prove that large $k \in \mathbb{Z}$, are $(\tau, h(k), \delta)$ regular for some τ, h, δ.

Lemma 2.5.10. *Suppose* $H\phi = E\phi$ *and* ϕ *satisfies* (2.4.2) *and* (2.4.3)*. Then* 0 *is* (τ, k, δ) *singular for any* $\tau, \delta > 0$.

Proof. It follows from (2.5.2) immediately. □

Thus all we need is to show that (τ, k, δ) singular points are far apart in a certain sense.

2.6. Cocycles and Lyapunov exponents By a cocycle, we mean a pair (T, A), where $T : \Omega \to \Omega$ is ergodic, A is a measurable 2×2 matrix valued function on Ω and $\det A = 1$.

We can regard it as a dynamical system on $\Omega \times \mathbb{R}^2$ with

$$(T, A) : (x, f) \longmapsto (Tx, A(x)f),\ (x, f) \in \Omega \times \mathbb{R}^2.$$

For $k > 0$, we define the k-step transfer matrix as

$$A_k(x) = \prod_{l=k}^{1} A(T^{l-1}x).$$

For $k < 0$, define $A_k(x) = A^{-1}_{-k}(T^k x)$. Denote $A_0 = I$, where I is the 2×2 identity matrix. Then $f_k(x) = \ln \|A_k(x)\|$ is a subadditive ergodic process. The (non-negative) Lyapunov exponent for the cocycle (α, A) is given by

$$L(T, A) = \inf_n \frac{1}{n} \int_\Omega \ln \|A_n(x)\| dx \overset{\text{a.e. } x}{=\!=\!=} \lim_{n\to\infty} \frac{1}{n} \ln \|A_n(x)\| dx. \tag{2.6.1}$$

with both the existence and the second equality in (2.6.1) guaranteed by Kingman's subadditive ergodic theorem. Cocycles with positive Lyapunov exponent are called hyperbolic. Here one should distinguish uniform hyperbolicity where there exists a continuous splitting of $\mathbb{R}^2$ into expanding and contracting directions, and nonuniform, where $L > 0$ but such splitting does not exist. Nevertheless,

Theorem 2.6.2 (Osceledec)**.** *Suppose* $L(T, A) > 0$. *Then, for almost every* $x \in \Omega$ *, there exist solutions* $v^+, v^- \in \mathbb{C}^2$ *such that* $\|A_k(x)v^\pm\|$ *decays exponentially at* $\pm\infty$*, respectively, at the rate* $-L(T, A)$*. Moreover, for every vector* w *which is linearly independent with* v^+ (*resp.,* v^-)*,* $\|A_k(x)w\|$ *grows exponentially at* $+\infty$ (*resp.,* $-\infty$) *at the rate* $L(T, A)$.

Suppose u is an eigensolution of $H_x u = Eu$. Then

$$\begin{bmatrix} u(n+m) \\ u(n+m-1) \end{bmatrix} = A_n(T^m x) \begin{bmatrix} u(m) \\ u(m-1) \end{bmatrix}, \tag{2.6.3}$$

where $A_n(x)$ is the transfer matrix of $A(x)$ and

$$A(x) = \begin{bmatrix} E - v(x) & -1 \\ 1 & 0 \end{bmatrix}.$$

Such $(T, A(x))$ is called the Schrödinger cocycle. Denote by $L(E)$ the Lyapunov exponent of the Schrödinger cocycle (we omit the dependence on T and v). It turns out that (at least for uniquely ergodic dynamics) the resolvent set of H is preciely the set of uniform hyperbolicity of the Schrödinger cocycle. The set $\sigma \cap \{L(E) > 0\}$ is therefore the set of non-uniform hyperbolicity, and is our main interest. Then Osceledec theorem can be reformulated as

Theorem 2.6.4. *Suppose that* $L(E) > 0$*. Then, for every* $x \in \Omega_E$ (Ω_E *has full measure), there exist solutions* ϕ^+, ϕ^- *of* $H_x\phi = E\phi$ *such that* $\phi^\pm$ *decays exponentially at* $\pm\infty$*, respectively, at the rate* $-L(E)$*. Moreover, every solution which is linearly independent of* ϕ^+ (*resp.,* ϕ^-) *grows exponentially at* $+\infty$ (*resp.,* $-\infty$) *at the rate* $L(E)$.

It turns out that the set where the Lyapunov exponent vanishes fully determines the absolutely continuous spectrum.

Theorem 2.6.5 (Ishii-Pastur-Kotani)**.** $\sigma_{ac}(H_x) = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{ess}$ *for almost every* $x \in \Omega$*.*

The inclusion "$\subseteq$" was proved by Ishii and Pastur [23, 40]. The other inclusion was proved by Kotani [35, 43]. Here we give a proof of the Ishii-Pastur part.

Proof. Denote $\mathcal{Z} = \{E \in \mathbb{R} : L(E) = 0\}$. If $L(E) > 0$, Oseledec' Theorem says that for almost every x, the eigensolution $u(x, E)$ of $H_x u = Eu$ is either exponentially decaying or exponentially growing. Applying Fubini's theorem, we see that for almost every x (with respect to P), the set of $E \in \mathbb{R} \setminus \mathcal{Z}$ for which the property just described fails, has zero Lebesgue measure. In other words, let $S_1 \subset \mathbb{R} \setminus \mathcal{Z}$ be the set with the non-Osceledec behavior. Then S_1 has zero Lebesgue measure. It implies that S_1 has zero weight with respect to the absolutely continuous part of any spectral measure. Let $S_2 \subset \mathbb{R} \setminus \mathcal{Z}$ be the set with the Osceledec behavior. In order to prove the Theorem, it suffices to show S_2 has zero weight with respect to any ac spectral measure. Indeed, if the solution of $H_x u = Eu$ is exponentially growing at ∞ or $-\infty$, by Schnol's theorem, such E does not make any contribution to the spectral measure. If the solution of $H_x u = Eu$ is exponentially decaying at both ∞ and $-\infty$, then E is an eigenvalue. The collection of eigenvalues must be countable, which also gives zero weight with respect to the ac spectral measure. □

It may seem that positive Lyapunov exponent should imply pure point spectrum with exponentially localized eigenfunctions, since, as above, for every E and a.e. phase a solution, if polynomially bounded, must decay exponentially on both sides.

2.7. Example: The Almost Mathieu Operator The almost Mathieu operator (AMO) is the (discrete) quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta} u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi(\theta + n\alpha) u(n), \tag{2.7.1}$$

where λ is the coupling, α is the frequency, and θ is the phase.

For the AMO, $L(E)$ can be computed exactly for E on the spectrum, but for now we will just need an estimate $L(E) \geqslant \ln \lambda$ for all $\alpha \notin \mathbb{Q}, E$ (See Theorem 3.0.2 for details). Thus, for $\lambda > 1$, Lyapunov exponent is stricly positive on the spectrum. In fact, we will later see that it does not even feel the arithmetics and is constant in the spectrum in both E and α.

We now quickly review the basics of continued fraction approximations.

2.8. Continued fraction expansion Define, as usual, for $0 \leqslant \alpha < 1$,

$$a_0 = 0, \alpha_0 = \alpha,$$

and, inductively for $k > 0$,

$$a_k = [\alpha_{k-1}^{-1}], \alpha_k = \alpha_{k-1}^{-1} - a_k.$$

We define

$$p_0 = 0, \quad q_0 = 1,$$
$$p_1 = 1, \quad q_1 = a_1,$$

and inductively,

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of best rational approximants to irrational number α, since it satisifies

$$\text{for any } 1 \leqslant k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}}. \tag{2.8.1}$$

Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \leqslant \Delta_n \triangleq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{1}{q_{n+1}}. \tag{2.8.2}$$

- α is called Diophantine if there exists $\kappa, \nu > 0$ such that $\|k\alpha\| \geqslant \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$, where $\|x\| = \min_{k\in\mathbb{Z}} |x - k|$.
- α is called Liouville if

$$\beta(\alpha) = \limsup_{k\to\infty} \frac{-\ln\|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n} > 0 \tag{2.8.3}$$

- α is called weakly Diophantine if $\beta(\alpha) = 0$.

Clearly, Diophantine implies weakly Diophantine. By Borel-Cantelli lemma, Diophantine α form a set of full Lebesgue measure.

Lemma 2.8.4 (Gordon [17], Simon [42])**.** *Suppose* $v \in C^1(\mathbb{T})$*. There is some constant* C *such that if* $\beta(\alpha) > C$*, then* $\sigma_{pp}(H_{v,\alpha,\theta}) = \emptyset$*.*

Remark: The constant in Lemma 2.8.4 can be estimated in a sharp way [8].

Lemma 2.8.4 is the first indication of the role of arithmetics in the spectral theory of quasiperiodic operators in the regime of positive LE, as it demonstrates the necessity of imposing an arithmetic condition.

Let us now denote

$$P_k(x) = \det(R_{[0,k-1]}(H_x - E)R_{[0,k-1]}).$$

It is easy to check by induction that

$$A_k(x) = \begin{pmatrix} P_k(x) & -P_{k-1}(Tx) \\ P_{k-1}(x) & -P_{k-2}(Tx) \end{pmatrix}. \tag{2.8.5}$$

Thus in the regime of positive $L(E)$, P_k "typically" behaves as $e^{kL(E)}$.

By Cramer's rule for given x_1 and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$|G_I(x_1, y)| = \left|\frac{P_{x_2-y}(T^{y+1}x)}{P_k(T^{x_1}x)}\right|, \tag{2.8.6}$$

$$|G_I(y, x_2)| = \left|\frac{P_{y-x_1}(T^{x_1}x)}{P_k(T^{x_1}x)}\right|. \tag{2.8.7}$$

Thus if P_k indeed hadn't deviated much from $e^{kL(E)}$ we would immediately have exponential decay of both terms. It turns out that for uniquely ergodic T there are no bad deviations for the numerator.

Lemma 2.8.8 ([14])**.** *Suppose* T *is uniquely ergodic, continuous and* A *is continuous. Then*

$$L(T, A) = \lim_{n\to\infty} \sup_{x\in\Omega} \frac{1}{n} \ln \|A_n(x)\|. \tag{2.8.9}$$

Under the assumptions of Lemma 2.8.8, we have for $\varepsilon > 0$,

$$|P_k(\theta)|, \|A_k(x)\| \leqslant e^{(L+\varepsilon)k}, \text{for } k \text{ large enough.} \tag{2.8.10}$$

Thus all deviations can only happen on the lower side. Let $A_{k,\epsilon} = \{x : |P_k(x)| < \exp((k+1)(L-\epsilon)\}$ be the large deviation set.

Lemma 2.8.11. *Assume* x *is* $(L-\epsilon, k, \frac{1}{4})$*-singular. Then, for large* k*, for some* $j \in I_{k,x} = [x - 3k/4, x - k/4]$ *we have* $T^{j+k-1}x \notin A_{k,\frac{1}{4}\epsilon+\epsilon_1}$ *for any* $\epsilon_1 > 0$*.*

Thus, two $(L-\epsilon, k, \frac{1}{4})$-singular points x_1, x_2 such that I_{k,x_1} and I_{k,x_2} do not intersect, produce two long strings of consecutive iterations that fall into the large deviation set.

3. Basics for the Almost Mathieu Operators

It is easy to see that $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k-1)\alpha$ and can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$:

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^j 2\pi(\theta + \frac{1}{2}(k-1)\alpha) \triangleq Q_k(\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)),$$

where Q_k is an algebraic polynomial of degree k.

For the almost Mathieu operator, the transfer matrix is given by

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha)\cdots A(\theta) \tag{3.0.1}$$

and $A(\theta) = \begin{bmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{bmatrix}$.

By Herman's trick [11, 19], we have the following lower bound estimates for $\lambda > 1$,

Theorem 3.0.2.

$$\int_{\mathbb{T}} (\ln |P_k|) d\theta \geqslant k \ln \lambda; \int_{\mathbb{T}} (\ln \|A_k\|) d\theta \geqslant k \ln \lambda. \tag{3.0.3}$$

For the AMO, the Lyapunov exponent on the spectrum actually can be obtained explicitly.

Theorem 3.0.4. [12]*For every* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \in \mathbb{R}$ *and* $E \in \sigma(H_{\lambda,\alpha,\theta})$, *one has* $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$. *Moreover, one can even compute the Lyapunov exponent* L^{ϵ} *of a complexified cocycle* $A(x+i\epsilon)$

It leads to the following three cases (see [2,3] for more general definitions).

Subcritical: $\lambda < 1$. In this case, it can be shown that $L^{\epsilon}(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon \leqslant \frac{-\ln\lambda}{2\pi}$. $H_{\lambda,\alpha,\theta}$ has purely ac spectrum [1,5].

Critical: $\lambda = 1$. In this case, it can be shown that $L(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$, but $L^{\epsilon}(E) > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon > 0$. $H_{\lambda,\alpha,\theta}$ has purely sc spectrum except possibly $\beta(\alpha) = 0$ and $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ [6,7,37].

Supercritical: $\lambda > 1$. $L(E) = \ln\lambda > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$.

In these lectures, we are interested only in the supercritical regime, $\lambda > 1$. In the following we always assume $E \in \sigma(H_{\lambda,\alpha,\theta})$.

The fact that $P_k(\theta) = Q_k(\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$ is a polynomial in $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha$ allows the use of the following Lagrange interpolation trick. Note that by Lagrange interpolation, $Q_k(x) = \sum_{j=1}^{k} \prod_{i\neq j} Q_k(x_j)\frac{x-x_i}{x_j-x_i}$. Thus if $\theta_i, i = 1, ..., k+1$, are in the large deviation set, we must have for some i,

$$\max_{x\in[-1,1]} \prod_{j=1, j\neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} > e^{k\epsilon}.$$

This motivates

Definition 3.0.5. We say that the set $\{\theta_1, \cdots, \theta_{k+1}\}$ is ϵ-uniform if

$$\max_{x\in[-1,1]} \max_{i=1,\cdots,k+1} \prod_{j=1,j\neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leqslant e^{k\epsilon}. \tag{3.0.6}$$

This is a convenient form of stating that θ_i have low discrepancy since $\int \ln|a - \cos 2\pi x| dx = -\ln 2$ for any $a \in [-1, 1]$.

We have the following Lemma.

Lemma 3.0.7. *Suppose* $\{\theta_1, \cdots, \theta_{k+1}\}$ *is* ϵ_1*-uniform. Then there exists a* θ_i *in the set* $\{\theta_1, \cdots, \theta_{k+1}\}$ *such that* $\theta_i - \frac{k-1}{2}\alpha \notin A_{k,\epsilon}$, *if* $\epsilon > \epsilon_1$ *and* k *is sufficiently large.*

We also have

Lemma 3.0.8. [4, Lemma 9.7] *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ *and* $0 \leqslant \ell_0 \leqslant q_n - 1$ *be such that* $|\sin\pi(x + \ell_0\alpha)| = \inf_{0\leqslant\ell\leqslant q_n-1} |\sin\pi(x + \ell\alpha)|$, *then for some absolute constant* $C > 0$,

$$-C\ln q_n \leqslant \sum_{\ell=0,\ell\neq\ell_0}^{q_n-1} \ln|\sin\pi(x+\ell\alpha)| + (q_n - 1)\ln 2 \leqslant C\ln q_n. \tag{3.0.9}$$

4. First transition line for Diophantine frequencies and phases

We already know that non-Diophantine α are trouble for localization, so let's fix a Diophantine α. It turns out, somewhat surprisingly, that θ matters as well.

- θ is called Diophantine with respect to α (sometimes we call $\alpha-$Diophantine) if there exists $\kappa, \nu > 0$ such that $\|2\theta + k\alpha\| \geqslant \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$.
- θ is called Liouville with respect to α if

$$\delta(\alpha, \theta) = \limsup_{k\to\infty} \frac{-\ln \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} > 0$$

- θ is called weakly Diophantine with respect to α if $\delta(\alpha, \theta) = 0$.

By Borel-Cantelli lemma, for fixed α, the set of $\alpha-$Diophantine has full Lebesgue measure. We have

Lemma 4.0.1 (J.-Simon [31])**.** *For even functions $v \in C^1(\mathbb{T})$, there exists some constant $C > 0$ such that if $\delta(\alpha, \theta) > C$, then $\sigma_{pp}(H_{v,\alpha,\theta}) = \emptyset$.*

Thus we need Diophantine-type conditions on both α and θ. In this section, we will prove

Theorem 4.0.2. *Suppose α is Diophantine and θ is Diophantine with respect to α. Then the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization.*

Remark 4.0.3.
- Theorem 4.0.2 was proved in [34]. Here the frame of the proof follows [34], with some modifications from [27,39].
- By the Aubry duality, it implies that for almost every α and θ, $H_{\lambda,\alpha,\theta}$ has purely ac spectrum [18,34].
- Actually, the proof of Theorem 4.0.2 holds for both weakly Diophantine frequencies and phases.

Suppose E is a generalized eigenvalue and ϕ is the corresponding generalized eigenfunction. Without loss of generality, assume $\phi(0) = 1$ (sometimes we assume $\phi^2(0) + \phi^2(1) = 1$). Take $k > 0$. Let n be such that $q_n \leqslant \frac{k}{4} < q_{n+1}$. Set I_1, I_2 as follows:

$$I_1 = [-q_n, q_n - 1] \tag{4.0.4}$$

and

$$I_2 = [k - q_n, k + q_n - 1]. \tag{4.0.5}$$

The set $\{\theta_j\}_{j\in I_1\cup I_2}$ consists of $4q_n$ elements, where $\theta_j = \theta + j\alpha$ and j range through $I_1 \cup I_2$.

Since α is Diophantine, one has

$$q_{n+1} \leqslant k^C, k \leqslant q_n^C.$$

Theorem 4.0.6. *For any $\varepsilon > 0$, the set $\{\theta_j\}_{j\in I_1\cup I_2}$ is ε-uniform if n is sufficiently large.*

Proof. We first estimate numerator in (3.0.6). In (3.0.6), let $x = \cos 2\pi a$ and take the logarithm. One has

$$\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$$

$$= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a+\theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a-\theta_j)| + (4sq_n - 1)\ln 2$$

$$= \Sigma_+ + \Sigma_- + (4q_n - 1)\ln 2, \tag{4.0.7}$$

where

$$\Sigma_+ = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a+\theta_j)|, \tag{4.0.8}$$

and

$$\Sigma_- = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a-\theta_j)|. \tag{4.0.9}$$

Both Σ_+ and Σ_- consist of 4 terms of the form of (3.0.9), plus 4 terms of the form

$$\ln \min_{j=0,1,\cdots,q_n-1} |\sin\pi(x+j\alpha)|, \tag{4.0.10}$$

minus $\ln|\sin\pi(a\pm\theta_i)|$. Since there exists an interval of length q_n in sum of (4.0.8) (or (4.0.9)) containing i, thus the minimum over this interval is not more than $\ln|\sin\pi(a\pm\theta_i)|$ (by the minimality). Thus, using (3.0.9) 4 times for Σ_+ and Σ_- respectively, one has

$$\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leqslant -4q_n\ln 2 + C\ln q_n. \tag{4.0.11}$$

The estimate of the denominator of (3.0.6) requires a bit more work. Without loss of generality, assume $i\in I_1$.

In (4.0.7), let $a=\theta_i$. We obtain

$$\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|$$

$$= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(\theta_i+\theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(\theta_i-\theta_j)| + (4q_n - 1)\ln 2$$

$$= \Sigma_+ + \Sigma_- + (4q_n - 1)\ln 2, \tag{4.0.12}$$

where

$$\Sigma_+ = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(2\theta + (i+j)\alpha)|, \tag{4.0.13}$$

and

$$\Sigma_- = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(i-j)\alpha|. \tag{4.0.14}$$

We will estimate Σ_+.

$I_1\cup I_2$ can be represented as a disjoint union of four segments B_j, each of length q_n. Applying (3.0.9) to each B_j, we obtain

$$\Sigma_+ > -4q_n\ln 2 + \sum_{j\in J_1\cup J_2} \ln|\sin\pi\hat{\theta}_j| - C\ln q_n - \ln|\sin 2\pi(\theta + i\alpha)|, \tag{4.0.15}$$

where

$$|\sin\pi\hat\theta_j| = \min_{\ell\in B_j}|\sin\pi(2\theta+(\ell+i)\alpha)|. \tag{4.0.16}$$

By the fact that θ is Diophantine with respect to α, we have

$$\ln|\sin\pi\hat\theta_j| \geqslant -C\ln|k| \geqslant -C\ln q_n. \tag{4.0.17}$$

Putting (4.0.15), (4.0.16) and (4.0.17) together, we have

$$\Sigma_+ > -4q_n\ln 2 - C\ln q_n. \tag{4.0.18}$$

Now let us estimate Σ_-. By the fact that α is Diophantine , we have for $i\neq j$, and $i,j\in I_1\cup I_2$,

$$\ln|\sin\pi(\theta_i-\theta_j)| \geqslant \ln|k|^{-C} \geqslant -C\ln q_n. \tag{4.0.19}$$

Replacing (4.0.17) with (4.0.19) and by the same argument as for Σ_+ , we have a similar estimate,

$$\Sigma_- > -4q_n\ln 2 - C\ln q_n. \tag{4.0.20}$$

From (4.0.12), (4.0.18) and (4.0.20), we have for any $\varepsilon>0$,

$$\max_{i\in I_1\cup I_2}\prod_{j\in I_1\cup I_2, j\neq i}\frac{|\cos 2\pi a-\cos 2\pi\theta_j|}{|\cos 2\pi\theta_i-\cos 2\pi\theta_j|} < e^{(4q_n-1)\varepsilon}, \tag{4.0.21}$$

for n large enough. □

Theorem 4.0.22. *Fix any $\varepsilon>0$. For any large $k\in\mathbb{Z}$, k is $(\ln\lambda-\varepsilon, y, \frac14)$ regular for some $k^{\frac1C}\leqslant y\leqslant k$.*

Proof. Define I_1 and I_2 as in (4.0.4) and (4.0.5). Take $y=4q_n$. Obviously, $k^{\frac1C}\leqslant y\leqslant k$. By Lemma 3.0.7, there exists some j_0 with $j_0\in I_1\cup I_2$ such that

$$\theta_{j_0}-\frac{4q_n-1}{2}\alpha\notin A_{4q_n-1,\varepsilon}.$$

By Lemma 2.8.11, for all $j\in I_1$, $\theta_j-\frac{4q_n-1}{2}\alpha\notin A_{4q_n-1,\varepsilon}$. Thus we have $j_0\in I_2$.

Set $I=[j_0-2q_n+1, j_0+2q_n-1]=[x_1,x_2]$. By (2.8.7), (2.8.7) and (2.8.10), it is easy to verify

$$|G_I(k,x_i)| \leqslant \exp\{(\ln\lambda+\varepsilon)(4q_n-1-|k-x_i|)-4q_n(\ln\lambda-\varepsilon)\}.$$

Notice that $|k-x_i|\geqslant q_n$, so we obtain

$$|G_I(k,x_i)| \leqslant \exp\{-(\ln\lambda-\varepsilon)|k-x_i|\}. \tag{4.0.23}$$

□

Proof of theorem 4.0.2. It follows from Theorems 2.5.3 and Theorem 4.0.22. □

5. Asymptotics of the eigenfunctions and proof of the second spectral transition line conjecture

By Theorem 2.6.5, $H_{\lambda,\alpha,\theta}$ does not have ac spectrum for $\lambda>1$. Lemmas 2.8.4 and 4.0.1 imply that $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $\delta(\alpha,\theta)$ or

$\beta(\alpha)$ is large, and we proved that there is Anderson localization if $\beta = \delta = 0$. Is there a sharp transition? The reason large β or δ are trouble is because they lead to resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. Indeed, large β leads to almost repetitions of the potential, and large δ to almost reflections. In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured in 1994 [24] that for the almost Mathieu family the two above types of resonances are the only ones that appear, and the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way.

Conjecture 1:

1a: (Diophantine phase) $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\lambda > e^{\beta(\alpha)}$ and $\delta(\alpha,\theta) = 0$, and $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all θ if $1 < \lambda < e^{\beta(\alpha)}$.

1b: (Diophantine frequency) Suppose $\beta(\alpha) = 0$. $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\lambda > e^{\delta(\alpha,\theta)}$, and has purely singular continuous spectrum if $1 < \lambda < e^{\delta(\alpha,\theta)}$.

1a says that without phase resonances, if the Lyapunov exponent beats the frequency resonance, then Anderson localization follows. Otherwise, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum. 1b says that without frequency resonances, if the Lyapunov exponent beats the phase resonance, then Anderson localization follows. Otherwise, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum.

In order to simplify the presentation, we assume

$$\lim_{n\to\infty} \frac{\ln q_{n+1}}{q_n} = \beta(\alpha). \tag{5.0.1}$$

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way. Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \leqslant k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows: for $\ell \geqslant 1$, let

$$\bar{r}_\ell^n = e^{-(\ln\lambda - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience. If $\ell q_n \leqslant k < (\ell+1)q_n$ with $\ell \geqslant 0$, set

$$f(k) = e^{-|k-\ell q_n|\ln\lambda}\bar{r}_\ell^n + e^{-|k-(\ell+1)q_n|\ln\lambda}\bar{r}_{\ell+1}^n, \tag{5.0.2}$$

and

$$g(k) = e^{-|k-\ell q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_{\ell+1}^n}. \tag{5.0.3}$$

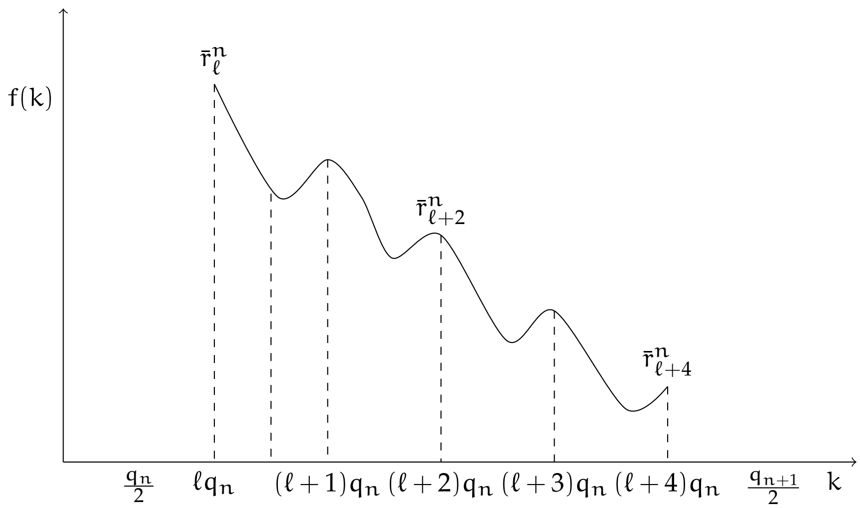


FIGURE 5.0.4. Graph of $f(k)$.

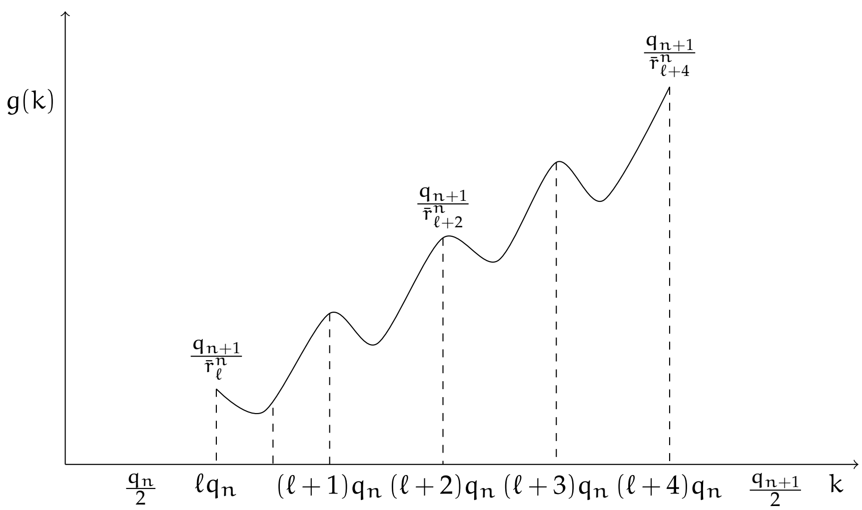


FIGURE 5.0.5. Graph of $g(k)$.

Theorem 5.0.6. [27] *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\lambda > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$ and Diophantine constants κ, ν) such that for any $|k| \geqslant K$, $U(k)$ and A_k satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leqslant \|U(k)\| \leqslant f(|k|)e^{\varepsilon|k|}, \tag{5.0.7}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leqslant \|A_k\| \leqslant g(|k|)e^{\varepsilon|k|}. \tag{5.0.8}$$

By (2.8.3), Theorem 5.0.6 implies the following Theorem.

Theorem 5.0.9. [27]*Suppose θ is Diophantine with respect to α. Then*

1. $H_{\lambda,\alpha,\theta}$ *has Anderson localization if* $\lambda > e^{\beta(\alpha)}$.
2. $H_{\lambda,\alpha,\theta}$ *has purely singular continuous spectrum if* $1 < \lambda < e^{\beta(\alpha)}$.
3. $H_{\lambda,\alpha,\theta}$ *has purely abolutely continuous spectrum if* $\lambda < 1$.

Remark 5.0.10. (1) Part 2 of Theorem 5.0.6 holds for $\delta(\alpha,\theta) = 0$.

(2) Part 2 is known for all α,θ [1] and is included here for completeness.
(3) Part 3 is known for all α,θ [8] and is included here for completeness.
(4) Parts 1 and 2 of Theorem 5.0.6 verify the frequency half of the conjecture in [24], as stated there. The measure theoretic version of the conjecture was proved in [8,25].

Corollary 5.0.11. *Under the condition of Theorem 5.0.6, we have*

I)
$$\limsup_{k\to\infty} \frac{\ln \|A_k\|}{k} = \ln\lambda,$$
II)
$$\liminf_{k\to\infty} \frac{\ln \|A_k\|}{k} = \ln\lambda - \beta.$$
III)
$$\limsup_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln\lambda,$$
IV)
$$\liminf_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln\lambda - \beta.$$

Now let us move to the Diophantine frequency case.

Theorem 5.0.12. [28] *Suppose α is Diophantine. We have*

1. $H_{\lambda,\alpha,\theta}$ *has Anderson localization if* $\lambda > e^{\delta(\alpha,\theta)}$.
2. $H_{\lambda,\alpha,\theta}$ *has purely singular continuous spectrum if* $1 < \lambda < e^{\delta(\alpha,\theta)}$.
3. $H_{\lambda,\alpha,\theta}$ *has purely abolutely continuous spectrum if* $\lambda < 1$.

Remark

(1) Parts 1 and 2 of Theorem 5.0.12 hold for weakly Diophantine α.
(2) We can prove part 2 for all irrational α, and general Lipshitz ν.
(3) Parts 1 and 2 of Theorem 5.0.12 verify the phase half of the conjecture in [24].

For the Diophantine frequencies case, we can also get the asymptotics of the eigenfunctions and transfer matricies. For simplicity, we only give the asymptotics of eigenfunctions. For any ℓ, let x_0 (we can choose any one if x_0 is not unique) be such that

$$|\sin\pi(2\theta + x_0\alpha)| = \min_{|x|\leqslant 2|\ell|} |\sin\pi(2\theta + x\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0\alpha \in \mathbb{Z}$, otherwise let $\eta \in (0, \infty)$ be given by the following equation,

$$|\sin \pi(2\theta + x_0\alpha)| = e^{-\eta|\ell|}. \tag{5.0.13}$$

Define $\hat{f} : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: $x_0 \cdot \ell \leqslant 0$. Set

$$\hat{f}(\ell) = e^{-|\ell|\ln\lambda}.$$

Case 2. $x_0 \cdot \ell > 0$. Set

$$\hat{f}(\ell) = e^{-(|x_0|+|\ell-x_0|)\ln|\lambda|}e^{\eta|\ell|} + e^{-|\ell|\ln\lambda}.$$

Theorem 5.0.14. [28] *Suppose* α *is Diophantine. Assume* $\ln\lambda > \delta(\alpha,\theta)$. *If* E *is a generalized eigenvalue and* ϕ *is the corresponding generalized eigenfunction of* $H_{\lambda,\alpha,\theta}$*, then for any* $\varepsilon > 0$*, there exists* K *such that for any* $|\ell| \geqslant K$, $U(\ell)$ *satisfies*

$$\hat{f}(\ell)e^{-\varepsilon|\ell|} \leqslant \|U(\ell)\| \leqslant \hat{f}(\ell)e^{\varepsilon|\ell|}. \tag{5.0.15}$$

6. Universal hierarchical structure for Diophantine phases and universal reflective-hierarchical structure for Diophantine frequencies

In this section, we will describe the universal hierarchical structure of the eigenfunctions in the Diophantine phase case. For Diophantine frequencies there is another, also universal, structure, conjectured to hold, for a.e. phase for all even functions, that features reflective-hierarchy. We refer the readers to [28] for the description of universal relective-hierarchical structure.

Note that Theorem 5.0.6 holds around arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions): $U(k)$ behaves as described at scale q_n but when seen in windows of size $q_k, q_k < q_{n-1}$ will demonstrate the same universal behavior around appropriate local maxima/minima.

To make the above precise, let ϕ be an eigenfunction, and $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$.

Let $I^j_{\sigma_1,\sigma_2} = [-\sigma_1 q_j, \sigma_2 q_j]$, for some $0 < \sigma_1, \sigma_2 \leqslant 1$. We will say k_0 is a local j-maximum of ϕ if $\|U(k_0)\| \geqslant \|U(k)\|$ for $k - k_0 \in I^j_{\sigma_1,\sigma_2}$. Occasionally, we will also use terminology (j,σ)-maximum for a local j-maximum on an interval $I^j_{\sigma,\sigma}$.

We will say a local j-maximum k_0 is *nonresonant* if

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{q_{j-1}{}^{\nu}},$$

for all $|k| \leqslant 2q_{j-1}$ and

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{6.0.1}$$

for all $2q_{j-1} < |k| \leqslant 2q_j$.

We will say a local j-maximum is *strongly nonresonant* if

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^\nu}, \tag{6.0.2}$$

for all $0 < |k| \leqslant 2q_j$.

An immediate corollary of Theorem 5.0.6 is the universality of behavior at all (strongly) nonresonant local maxima.

Theorem 6.0.3. *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a local j-maximum for $j > j(\epsilon)$, then the following two statements hold:*

If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leqslant \frac{\|U(k_0+s)\|}{\|U(k_0)\|} \leqslant f(|s|)e^{\varepsilon|s|}, \tag{6.0.4}$$

for all $2s \in I^j_{\sigma_1,\sigma_2}$, $|s| > \frac{q_{j-1}}{2}$.

If k_0 is strongly nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leqslant \frac{\|U(k_0+s)\|}{\|U(k_0)\|} \leqslant f(|s|)e^{\varepsilon|s|}, \tag{6.0.5}$$

for all $2s \in I^j_{\sigma_1,\sigma_2}$.

Theorem 5.0.6 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction. Let k_0 be a global maximum .

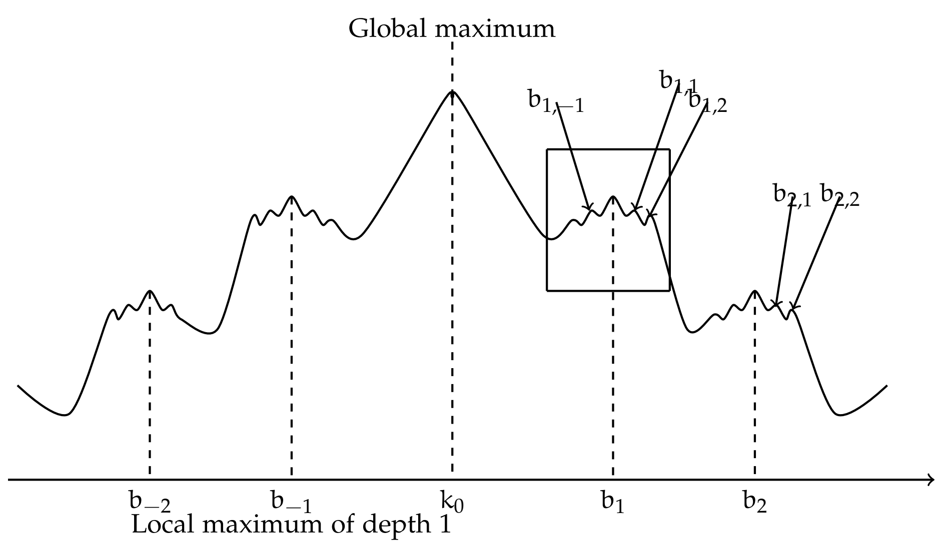


Fig.3

FIGURE 6.0.6. Universal hierarchical structure of an eigenfunction.

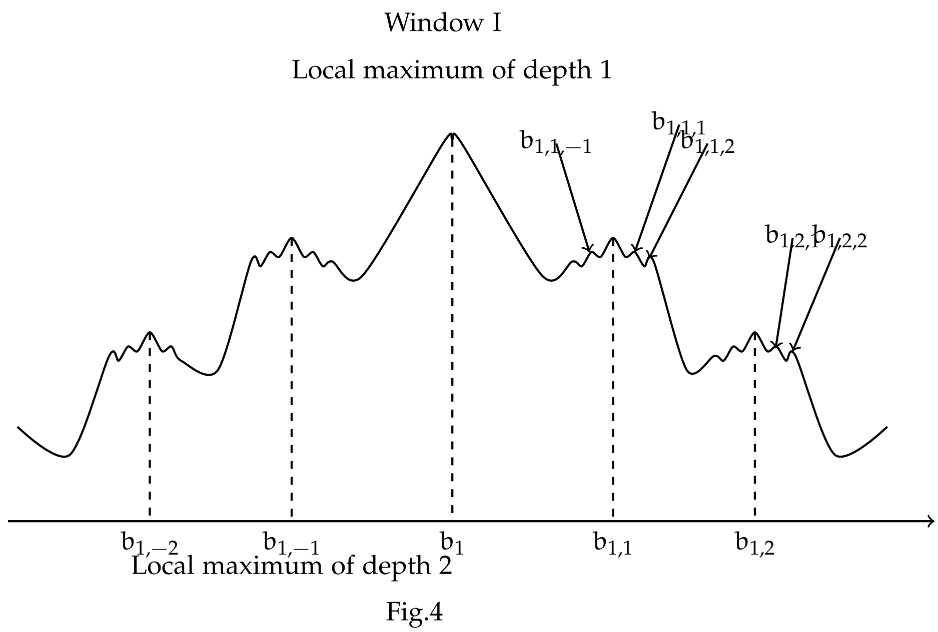


FIGURE 6.0.7. Universal hierarchical structure of an eigenfunction.

We first describe the hierarchical structure of local maxima informally. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a constant scale $\hat{n}_0$ thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$. The exponential behavior of the eigenfunction in the local neighborhood (of size $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible. Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$ we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each $0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$, by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$ we then get the same picture taking the magnifying glass another level deeper and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$, see Corollary **??**. Fig. 3 schematically illustrates the structure of local maxima of depth one and two, and Fig. 4 illustrates that the the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum.

We now describe the hierarchical structure precisely. Suppose

$$\|2(\theta + k_0\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{6.0.8}$$

for any $k \in \mathbb{Z}\backslash\{0\}$. Fix $0 < \sigma, \epsilon$ with $\sigma + 2\epsilon < 1$. Let $n_j \to \infty$ be such that $\ln q_{n_j+1} \geqslant (\sigma + 2\epsilon) \ln |\lambda| q_{n_j}$. Let $\mathfrak{c}_j = (\ln q_{n_j+1} - \ln |a_{n_j}|)/\ln |\lambda| q_{n_j} - \epsilon$. We have $\mathfrak{c}_j > \epsilon$ for $0 < a_{n_j} < e^{\sigma \ln |\lambda| q_{n_j}}$. Then we have

Theorem 6.0.9. *There exists $\hat{n}_0(\alpha, \lambda, \kappa, \nu, \epsilon) < \infty$ such that for any $j_0 > j_1 > \cdots > j_k$, $n_{j_k} \geqslant \hat{n}_0 + k$, and $0 < a_{n_{j_i}} < e^{\sigma \ln |\lambda| q_{n_{j_i}}}$, $i = 0, 1, \ldots, k$, for all $0 \leqslant s \leqslant k$ there exists a local n_{j_s}-maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ on the interval $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} + I^{n_{j_s}}_{\mathfrak{c}_{j_s}, 1}$ for all $0 \leqslant s \leqslant k$ such that the following holds:*

I: $|b_{a_{n_{j_0}}} - (k_0 + a_{n_{j_0}} q_{n_{j_0}})| \leqslant q_{\hat{n}_0+1}$,

II: *For any* $1 \leqslant s \leqslant k$, $|b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} - (b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + a_{n_{j_s}} q_{n_{j_s}})| \leqslant q_{\hat{n}_0+s+1}$.

III: *if* $2(x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}) \in I^{n_{j_k}}_{\mathfrak{c}_{j_k}, 1}$ *and* $|x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}| \geqslant q_{\hat{n}_0+k}$, *then for each* $s = 0, 1, \ldots, k$,

$$f(x_s) e^{-\varepsilon |x_s|} \leqslant \frac{\|U(x)\|}{\|U(b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}})\|} \leqslant f(x_s) e^{\varepsilon |x_s|}, \tag{6.0.10}$$

where $x_s = |x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}|$ *is large enough.*

Moreover, every local n_{j_s}-maximum on the interval $b_{a_{n_j}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + [-e^{\epsilon \ln \lambda q_{n_{j_s}}}, e^{\epsilon \ln \lambda q_{n_{j_s}}}]$ is of the form $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ for some $a_{n_{j_s}}$.

7. Proof of Theorem 5.0.6

Define $b_n = q_n^t$ with $\frac{8}{9} \leqslant t < 1$ (t will be defined later). For any $k > 0$, we will distinguish two cases with respect to n:

(i) $|k - \ell q_n| \leqslant b_n$ for some $\ell \geqslant 1$, called n*−resonance*.

(ii) $|k - \ell q_n| > b_n$ for all $\ell \geqslant 0$, called n*−nonresonance*.

Let s be the largest integer such that $4 s q_{n-1} \leqslant \text{dist}(y, q_n \mathbb{Z})$.

Theorem 7.0.1. *Assume $\lambda > e^{\beta(\alpha)}$ and θ is Diophantine with respect to α. Suppose either*

i) $b_n \leqslant |y| < C b_{n+1}$, *where $C > 1$ is a fixed constant*

or

ii) $0 \leqslant |y| < q_n$.

Then for any $\varepsilon > 0$ *and* n *large enough, if* y *is* $n-$*nonresonant, we have* y *is* $(\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} - \varepsilon, 4sq_{n-1} - 1, \frac{1}{4})$ *regular.*

Proof. We again assume for simplicity $\lim \frac{\ln q_{n+1}}{q_n} = \beta(\alpha)$. Then since $\lim \frac{\ln q_{n+1}}{q_n} = \beta(\alpha) > 0$, we have $s > 0$ for large n. For an $n-$nonresonant y in the Theorem, one has

$$\min_{j,i\in I_1\cup I_2} \ln|\sin\pi(2\theta + (j+i)\alpha)| \geqslant -C\ln q_n. \tag{7.0.2}$$

and

$$\min_{i\neq j; i,j\in I_1\cup I_2} \ln|\sin\pi(j-i)\alpha)| \geqslant -C\ln q_n. \tag{7.0.3}$$

The idea is similar to what we used in the proof of Theorem 4.0.6. We employ the same notations in Theorem 4.0.6.

The upper bound of $\sum_{j\in I_1\cup I_2, j\neq i}\ln|\cos 2\pi a - \cos 2\pi\theta_j|$ is the same as (4.0.11). (4.0.12)-(4.0.14) also hold. However the estimate of $\sum_{j\in J_1}\ln|\sin\pi\hat{\theta}_j|$ is much more difficult in the non-Diophantine case. Here we sketch the argument.

Assume that $\hat{\theta}_{j+1} = \hat{\theta}_j + q_n\alpha$ for every $j, j+1 \in J_1$. Applying the Stirling formula and (7.0.2), one has

$$\begin{aligned}\sum_{j\in J_1}\ln|\sin 2\pi\hat{\theta}_j| &> 2\sum_{j=1}^{s}\ln\frac{j\Delta_n}{C} - C\ln q_n \\ &> 2s\ln\frac{s}{q_{n+1}} - Cs\ln q_n.\end{aligned} \tag{7.0.4}$$

In the other cases, decompose J_1 in maximal intervals T_κ such that for $j, j+1 \in T_\kappa$ we have $\hat{\theta}_{j+1} = \hat{\theta}_j + q_n\alpha$. Notice that the boundary points of an interval T_κ are either boundary points of J_1 or satisfy $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_n \geqslant \frac{\Delta_{n-1}}{2}$. This follows from the fact that if $0 < |z| < q_n$, then $\|\hat{\theta}_j + q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leqslant \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_n$, and $\|\hat{\theta}_j + (z+q_n)\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \|z\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|\hat{\theta}_j + q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \Delta_{n-1} - \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} - \Delta_n$. Assuming $T_\kappa \neq J_1$, then there exists $j \in T_\kappa$ such that $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} \geqslant \frac{\Delta_{n-1}}{2} - \Delta_n$.

If T_κ contains some j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-1}}{10}$, then

$$\begin{aligned}|T_\kappa| &\geqslant \frac{\frac{\Delta_{n-1}}{2} - \Delta_n - \frac{\Delta_{n-1}}{10}}{\Delta_n} \\ &\geqslant \frac{1}{4}\frac{\Delta_{n-1}}{\Delta_n} - 1 \geqslant s-1,\end{aligned} \tag{7.0.5}$$

where $|T_\kappa| = b - a + 1$ for $T_\kappa = [a, b]$. For such T_κ, a similar estimate to (7.0.4) gives

$$\begin{aligned}\sum_{j\in T_\kappa}\ln|\sin\pi\hat{\theta}_j| &> |T_\kappa|\ln\frac{|T_\kappa|}{q_{n+1}} - Cs\ln q_n \\ &> |T_\kappa|\ln\frac{s}{q_{n+1}} - Cs\ln q_n.\end{aligned} \tag{7.0.6}$$

If T_κ does not contain any j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-1}}{10}$, then by (2.8.2)

$$\begin{aligned}\sum_{j\in T_\kappa} \ln|\sin \pi\hat{\theta}_j| &> -|T_\kappa|\ln q_n - C|T_\kappa| \\ &> |T_\kappa|\ln\frac{s}{q_{n+1}} - C|T_\kappa|. \qquad (7.0.7)\end{aligned}$$

By (7.0.6) and (7.0.7), one has

$$\sum_{j\in J_1} \ln|\sin \pi\hat{\theta}_j| \geqslant 2s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \qquad (7.0.8)$$

Similarly,

$$\sum_{j\in J_2} \ln|\sin \pi\hat{\theta}_j| \geqslant 2s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \qquad (7.0.9)$$

Putting (4.0.15), (7.0.8) and (7.0.9) together, we have

$$\Sigma_+ > -4sq_n\ln 2 + 6s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \qquad (7.0.10)$$

Now let us estimate Σ_-.

Replacing (7.0.2) with (7.0.3) and following the discussion of Σ_+ , we have the similar estimate,

$$\Sigma_- > -4sq_n\ln 2 + 4s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \qquad (7.0.11)$$

From (4.0.12), (7.0.10) and (7.0.11), it follows

$$\begin{aligned}&\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j| \\ &> -4sq_n\ln 2 + 8s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \qquad (7.0.12)\end{aligned}$$

Combining with (4.0.11), we have for any $\varepsilon > 0$,

(7.0.13)

$$\max_{i\in I_1\cup I_2}\prod_{j\in I_1\cup I_2, j\neq i}\frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leqslant e^{(4sq_n-1)(\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} - \varepsilon)}.$$

□

Remark 7.0.14. In the nonresonant case, for any $\varepsilon > 0, \frac{8}{9} \leqslant t < 1$, one has $\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} \geqslant \ln\lambda - 8(1-t)\beta - \varepsilon > 0$. In addition, we have $\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} \geqslant \ln\lambda - 2\varepsilon$ if t is close to 1.

Remark 7.0.15. Here, we only use Theorem 7.0.1 with $C = 50C_\star$, where $C_\star$ is given by (7.0.16) (see below).

Clearly, it is enough to consider $k > 0$. In this section we study the resonant case. Suppose there exists some $k \in [b_n, b_{n+1}]$ such that k is n−resonant. For any $\varepsilon > 0$, choose $\eta = \frac{\varepsilon}{C}$, where C is a large constant (depending on λ, α).

Let

$$C_* = 2(1 + \lfloor\frac{\ln\lambda}{\ln\lambda - \beta}\rfloor), \qquad (7.0.16)$$

where $\lfloor m \rfloor$ denotes the smallest integer not exceeding m.

For an arbitrary solution φ satisfying $H\varphi = E\varphi$, let

$$r_j^{n,\varphi} = \sup_{|r|\leqslant 10\eta} |\varphi(jq_n + rq_n)|,$$

where $|j| \leqslant 50C_* \frac{b_{n+1}}{q_n}$.

Let ϕ be the generalized eigenfunction. Denote by

$$r_j^n = r_j^{n,\phi}.$$

Since we keep n fixed in this section we omit the dependence on n from the notation and write r_j^φ, R_j, and r_j.

Note that below we always assume n is large enough.

Lemma 7.0.17. *Let* $k \in [jq_n, (j+1)q_n]$ *with* $\mathrm{dist}(k, q_n\mathbb{Z}) \geqslant 10\eta q_n$. *Suppose* $|j| \leqslant 48C_* \frac{b_{n+1}}{q_n}$. *Then for sufficiently large* n,

(7.0.18)
$$|\varphi(k)| \leqslant \max\{r_j^\varphi \exp\{-(\ln\lambda - 2\eta)(d_j - 3\eta q_n)\}, r_{j+1}^\varphi \exp\{-(\ln\lambda - 2\eta)(d_{j+1} - 3\eta q_n)\}\},$$

where $d_j = |k - jq_n|$ *and* $d_{j+1} = |k - jq_n - q_n|$.

Proof. The proof builds on the ideas akin to those used in the proof of Theorem 2.5.3. However it requires a more careful approach.

For any $y \in [jq_n + \eta q_n, (j+1)q_n - \eta q_n]$, apply i) of Theorem 7.0.1 with $C = 50C_\star$. Notice that in this case, we have

$$\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} - \eta \geqslant \ln\lambda - 2\eta.$$

Thus y is regular with $\tau = \ln\lambda - 2\eta$. Therefore there exists an interval $I(y) = [x_1, x_2] \subset [jq_n, (j+1)q_n]$ such that $y \in I(y)$ and

$$\mathrm{dist}(y, \partial I(y)) \geqslant \frac{1}{4}|I(y)| \geqslant q_{n-1} \tag{7.0.19}$$

and

$$|G_{I(y)}(y, x_i)| \leqslant e^{-(\ln\lambda - 2\eta)|y - x_i|},\ i = 1, 2, \tag{7.0.20}$$

where $\partial I(y)$ is the boundary of the interval $I(y)$, i.e.,$\{x_1, x_2\}$, and $|I(y)|$ is the size of $I(y) \cap \mathbb{Z}$, i.e., $|I(y)| = x_2 - x_1 + 1$. For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 \leqslant (j+1)q_n - \eta q_n$ or $x_1 - 1 \geqslant jq_n + \eta q_n$, we can expand $\varphi(x_2 + 1)$ or $\varphi(x_1 - 1)$ using (2.5.2). We can continue this process until we arrive to z such that $z + 1 > (j+1)q_n - \eta q_n$ or $z - 1 < jq_n + \eta q_n$, or the iterating number reaches $\lfloor \frac{2q_n}{q_{n-1}} \rfloor$. Thus, by (2.5.2)

(7.0.21)
$$\varphi(k) = \sum_{s; z_{i+1} \in \partial I(z_i')} G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}'),$$

where in each term of the summation one has $jq_n + \eta q_n + 1 \leqslant z_i \leqslant (j+1)q_n - \eta q_n - 1$, $i = 1, \cdots, s$, and either $z_{s+1} \notin [jq_n + \eta q_n + 1, (j+1)q_n - \eta q_n - 1]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$; or $s + 1 = \lfloor \frac{2q_n}{q_{n-1}} \rfloor$. We should mention that $z_{s+1} \in [jq_n, (j+1)q_n]$.

If $z_{s+1} \in [jq_n, jq_n + \eta q_n]$, $s+1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, this implies

$$|\varphi(z'_{s+1})| \leqslant r_j^{\varphi}.$$

By (7.0.20), we have

$$\begin{aligned} &|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\varphi(z'_{s+1})| \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda-2\eta)(|k-z_1|+\sum_{i=1}^{s}|z'_i-z_{i+1}|)} \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda-2\eta)(|k-z_{s+1}|-(s+1))} \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda-2\eta)(d_j-2\eta q_n-4-\frac{2q_n}{q_{n-1}})}. \end{aligned} \tag{7.0.22}$$

If $z_{s+1} \in [(j+1)q_n - \eta q_n, (j+1)q_n]$, $s+1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, by the same arguments, we have

$$|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\varphi(z'_{s+1})| \leqslant r_{j+1}^{\varphi} e^{-(\ln\lambda-2\eta)(d_{j+1}-2\eta q_n-4-\frac{2q_n}{q_{n-1}})}. \tag{7.0.23}$$

If $s+1 = \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, using (7.0.19) and (7.0.20), we obtain

$$|G_{I(k)}(k,z_1)G_{I(z'_1)}(z'_1,z_2)\cdots G_{I(z'_s)}(z'_s,z_{s+1})\varphi(z'_{s+1})| \leqslant e^{-(\ln\lambda-2\eta)q_{n-1}\lfloor \frac{2q_n}{q_{n-1}} \rfloor}|\varphi(z'_{s+1})|. \tag{7.0.24}$$

Notice that the total number of terms in (7.0.21) is at most $2^{\lfloor \frac{2q_n}{q_{n-1}} \rfloor}$ and $d_j, d_{j+1} \geqslant 10\eta q_n$. By (7.0.22), (7.0.23) and (7.0.24), we have

$$|\varphi(k)| \leqslant \max\{r_j^{\varphi} e^{-(\ln\lambda-2\eta)(d_j-3\eta q_n)}, r_{j+1}^{\varphi} e^{-(\ln\lambda-2\eta)(d_{j+1}-3\eta q_n)}, \max_{p\in[jq_n,(j+1)q_n]}\{e^{-(\ln\lambda-2\eta)q_n}|\varphi(p)|\}\}. \tag{7.0.25}$$

Now we will show that for any $p \in [jq_n, (j+1)q_n]$, one has $|\varphi(p)| \leqslant \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$. Then (7.0.25) implies case i) of Lemma 7.0.17. Otherwise, by the definition of r_j^{φ}, if $|\varphi(p')|$ is the largest one of $|\varphi(z)|, z \in [jq_n + 10\eta q_n + 1, (j+1)q_n - 10\eta q_n - 1]$, then $|\varphi(p')| > \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$. Applying (7.0.25) to $\varphi(p')$ and noticing that $\mathrm{dist}(p', q_n\mathbb{Z}) \geqslant 10\eta q_n$, we get

$$|\varphi(p')| \leqslant e^{-7(\ln\lambda-2\eta)\eta q_n}\max\{r_j^{\varphi}, r_{j+1}^{\varphi}, |\varphi(p')|\}.$$

This is impossible because $|\varphi(p')| > \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$.

□

By the properties of continued fractions and since θ is α-Diophantine, one can obtain

Lemma 7.0.26. *For any $|i|, |j| \leqslant 50C_* b_{n+1}$, the following estimate holds,*

$$\ln|\sin\pi(2\theta + (j+i)\alpha)| \geqslant -C\ln q_n. \tag{7.0.27}$$

and

Lemma 7.0.28. *Assume $|i|, |j| \leqslant 50C_* b_{n+1}$, and $i - j \neq q_n\mathbb{Z}$. Then*

$$\ln|\sin\pi(j-i)\alpha| \geqslant -C\ln q_n. \tag{7.0.29}$$

We then have

Theorem 7.0.30. *For* $1 \leqslant j \leqslant 46C_\star \frac{b_{n+1}}{q_n}$*, the following holds*

(7.0.31) $$r_j^\varphi \leqslant \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}\}.$$

Proof. Fix j with $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$ and $|r| \leqslant 10\eta q_n$. Set $I_1, I_2 \subset \mathbb{Z}$ as follows

$$\begin{aligned} I_1 &= [-\lfloor\frac{1}{2}q_n\rfloor, q_n - \lfloor\frac{1}{2}q_n\rfloor - 1], \\ I_2 &= [jq_n - \lfloor\frac{1}{2}q_n\rfloor, (j+1)q_n - \lfloor\frac{1}{2}q_n\rfloor - 1]. \end{aligned}$$

Let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m \in I_1 \cup I_2}$ consists of $2q_n$ elements.

By Lemmas 7.0.26 and 7.0.28, and following the proof of Theorem 4.0.6, one obtains that $\{\theta_m\}$ is $\frac{\ln q_{n+1} - \ln j}{2q_n} + \varepsilon$ uniform for any $\varepsilon > 0$. Combining with Lemma 3.0.7, there exists some j_0 with $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{2q_n - 1, \ln\lambda - \frac{\ln q_{n+1} - \ln j}{2q_n} - \eta}$.

First, we assume $j_0 \in I_2$.

Set $I = [j_0 - q_n + 1, j_0 + q_n - 1] = [x_1, x_2]$. In (2.8.10), let $\varepsilon = \eta$. Combining with (2.8.7) and (2.8.7), it is easy to verify

$$|G_I(jq_n + r, x_i)| \leqslant e^{(\ln\lambda + \eta)(2q_n - 1 - |jq_n + r - x_i|) - (2q_n - 1)(\ln\lambda - \frac{\ln q_{n+1} - \ln j}{2q_n} - \eta)}.$$

Using (2.5.2), we obtain

(7.0.32) $$|\varphi(jq_n + r)| \leqslant \sum_{i=1,2} \frac{q_{n+1}}{j} e^{5\eta q_n} |\varphi(x_i')| e^{-|jq_n + r - x_i|\ln\lambda},$$

where $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$.

Let $d_j^i = |x_i - jq_n|$, $i = 1, 2$. It is easy to check that

(7.0.33) $$|jq_n + r - x_i| + d_j^i, |jq_n + r - x_i| + d_{j\pm1}^i \geqslant q_n - |r|,$$

and

(7.0.34) $$|jq_n + r - x_i| + d_{j\pm2}^i \geqslant 2q_n - |r|.$$

If $\mathrm{dist}(x_i, q_n\mathbb{Z}) \geqslant 10\eta q_n$, then we bound $\varphi(x_i)$ in (7.0.32) using (7.0.18). If $\mathrm{dist}(x_i, q_n\mathbb{Z}) \leqslant 10\eta q_n$, then we bound $\varphi(x_i)$ in (7.0.32) by some proper r_j. Combining with (7.0.33), (7.0.34), we have

$$r_j^\varphi \leqslant \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}, r_j^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}, r_{j\pm2}^\varphi \frac{q_{n+1}}{j}\exp\{-2(\ln\lambda - C\eta)q_n\}\}.$$

However

$$\begin{aligned} r_j^\varphi &\leqslant r_j^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\} \\ &\leqslant r_j^\varphi \exp\{-(\ln\lambda - \beta - C\eta)q_n\} \end{aligned}$$

cannot happen, so we must have

(7.0.35)
$$r_j^\varphi \leqslant \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}, r_{j\pm2}^\varphi \frac{q_{n+1}}{j}\exp\{-2(\ln\lambda - C\eta)q_n\}\}.$$

In particular,

(7.0.36) $$r_j^\varphi \leqslant \exp\{-(\ln\lambda - \beta - C\eta)q_n\}\max\{r_{j\pm1}^\varphi\, r_{j\pm2}^\varphi\}.$$

If $j_0 \in I_1$, then (7.0.36) holds for $j = 0$. Let $\varphi = \phi$ in (7.0.36). We get

$$|\phi(0)|, |\phi(-1)| \leqslant \exp\{-(\ln\lambda - \beta - C\eta)q_n\},$$

which is in contradiction with $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Therefore $j_0 \in I_2$, so (7.0.35) holds for any φ.

By (2.6.3) and (2.8.10), we have

$$\left\| \begin{pmatrix} \varphi(k_1) \\ \varphi(k_1 - 1) \end{pmatrix} \right\| \geqslant Ce^{-(\ln\lambda+\varepsilon)|k_1-k_2|} \left\| \begin{pmatrix} \varphi(k_2) \\ \varphi(k_2 - 1) \end{pmatrix} \right\|. \tag{7.0.37}$$

This implies

$$r^{\varphi}_{j\pm 2} \leqslant r^{\varphi}_{j\pm 1} \exp\{(\ln\lambda + C\eta)q_n\},$$

thus (7.0.35) becomes

$$r^{\varphi}_j \leqslant \max\{r^{\varphi}_{j\pm 1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{7.0.38}$$

for any $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$. □

We now show that by Theorem 2.4.1 exponential growth is not allowed, r_j must actually decay.

Theorem 7.0.39. *For* $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$*, the following holds*

$$r_j \leqslant r_{j-1} \exp\{-(\ln\lambda - C\eta)q_n\} \frac{q_{n+1}}{j}. \tag{7.0.40}$$

Proof. Let $\varphi = \phi$ in Lemma 7.0.30. We must have

$$r_j \leqslant \max\{r_{j\pm 1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{7.0.41}$$

for any $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$.

Suppose for some $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$, the following holds,

$$r_j \leqslant r_{j+1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\} \leqslant r_{j+1} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{7.0.42}$$

Applying (7.0.41) to $j+1$, we obtain

$$r_{j+1} \leqslant \max\{r_j, r_{j+2}\} \frac{q_{n+1}}{j+1} \exp\{-(\ln\lambda - C\eta)q_n\}. \tag{7.0.43}$$

Combining with (7.0.42), we must have

$$r_{j+1} \leqslant r_{j+2} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{7.0.44}$$

Generally, for any $0 < p \leqslant (C_* + 1)j - 1$, we obtain

$$r_{j+p} \leqslant r_{j+p+1} \exp\{-(\ln\lambda - \beta - C\eta)q_n\}. \tag{7.0.45}$$

Thus

$$r_{(C_*+1)j} \geqslant r_j \exp\{(\ln\lambda - \beta - C\eta)C_* j q_n\}. \tag{7.0.46}$$

Clearly, by (7.0.37), one has

$$r_j \geqslant \exp\{-(\ln\lambda + C\eta)jq_n\}.$$

Then

$$r_{(C_*+1)j} \geqslant \exp\{((C_*-1)\ln\lambda - C_*\beta - C\eta)jq_n\}. \tag{7.0.47}$$

By the definition of C_*, one has

$$(C_*-1)\ln\lambda - C_*\beta > 0.$$

Thus (7.0.47) is in contradiction with the fact that $|\phi(k)| \leqslant 1+|k|$.

Now that (7.0.42) can not happen, from (7.0.41), we must have

$$r_j \leqslant r_{j-1}\frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}. \tag{7.0.48}$$

□

Theorem 7.0.49. *For $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$, the following holds*

$$r_j \geqslant r_{j-1}\exp\{-(\ln\lambda - \varepsilon)q_n\}\frac{q_{n+1}}{j}. \tag{7.0.50}$$

Proof. See [27] for details. □

We are now ready to complete the proof.

Proof of Theorem 5.0.6. Set $t_0 = 1 - \frac{\varepsilon}{8\beta}$. Let $t = t_0$ in the definition of resonance, i.e. $b_n = q_n^{t_0}$.

Case I: $\ell \geqslant q_{n+1}^{t_0}$

By case II of Theorem 7.0.1, for any $y \in (\varepsilon q_{n+1}^{t_0}, q_{n+1} - \varepsilon q_{n+1}^{t_0})$, y is $(\ln\lambda + 8\ln(sq_n/q_{n+1})/q_n - \varepsilon, 4sq_n - 1)$ regular with $\delta = \frac{1}{4}$. Notice that

$$(s+1)q_n \geqslant \varepsilon q_{n+1}^{t_0} \geqslant \varepsilon q_{n+1}^{t_0},$$

thus we have

$$\begin{aligned} \ln\lambda + 8\ln(sq_n/q_{n+1})/q_n &\geqslant \ln\lambda - 8(1-t_0)\beta - \varepsilon \\ &\geqslant \ln\lambda - 2\varepsilon. \end{aligned}$$

Thus for any $y \in (\varepsilon q_{n+1}^{t_0}, q_{n+1} - \varepsilon q_{n+1}^{t_0})$, y is $(\ln\lambda - 2\varepsilon, 4sq_n - 1)$ regular. Following the proof of Lemma 7.0.17, one has for $\ell q_n \leqslant k \leqslant (\ell+1)q_n$,

$$\|U(k)\| \leqslant e^{-(\ln\lambda - \varepsilon)|k|},$$

which implies Theorem 5.0.6 in this case.

Case II: $0 \leqslant \ell \leqslant q_{n+1}^{t_0}$

By Theorems 7.0.49 and 7.0.39, and Stirling formula,

$$\bar{r}_j^n e^{-\varepsilon j q_n} \leqslant r_j \leqslant \bar{r}_j^n e^{\varepsilon j q_n}.$$

Now Theorem 5.0.6 follows from Lemma 7.0.17.

□

8. Arithmetic criteria for spectral dimension

We know that in the regime of positive Lyapunov exponent the spectrum is always singular. Now that we also know (Lemma 2.8.4) that large β implies continuous (and therefore singular continuous) spectrum, it's natural to ask whether even larger β implies increased continuity. "Continuity" of singular continuous spectrum can be quantified through fractal dimensions. The most popular object is Hausdorff dimension. However Hausdorff dimension is a poor tool for characterizing the singular continuous spectral measures arising in the regime of positive Lyapunov exponents, as it is always equal to zero (a very general theorem of Simon that holds for general ergodic potentials and a.e. phase, see Theorem 8.2.6 [44] (and for every phase for the zero entropy dynamical systems [20] (see also [26, 30]). It turns out that some other dimensions do present good tools to finely distinguish between different kinds of singular continuous spectra appearing in the supercritical regime. The main goal of this lecture is to briefly present a statement that is simultaneously a quantitative version of two well known statements

1. *Periodicity implies absolute continuity.* We prove that a quantitative weakening (near periodicity that holds sufficiently long) implies quantitative continuity of the (fractal) spectral measure.

2. *Gordon condition (a single/double almost repetition) implies continuity of the spectral measure.* Indeed, we prove that a quantitative strengthening (multiple almost repetitions) implies quantitative continuity of the spectral measure.

This will allow us to establish a sharp arithmetic criterion for certain dimension of the spectral measure in terms of β, for *general* analytic potentials.

8.1. m-function and subordinacy theory Let μ be a finite Borel measure on $\mathbb{R}$. Define the Borel transform of μ to be:

$$m(z) := \int \frac{1}{E-z} d\mu(E), \quad z \in \mathbb{C}. \tag{8.1.1}$$

It is easy to check that for any finite Borel measure μ on $\mathbb{R}$, its m-function is holomorphic in the upper half plane and satisfies

$$m^*(z) = m(z^*), \quad |m(z)| \leqslant \frac{\mu(\mathbb{R})}{\operatorname{Im} z} \quad z \in \mathbb{C}_+.$$

Remark 8.1.2. Functions with this property are known as Herglotz, Pick or R functions. They map the upper half-plane into itself, but are not necessarily injective or surjective. m is holomorphic in $\mathbb{C} \backslash \sigma(\mu)$, where $\sigma(\mu) := \{E \in \mathbb{R} : \mu(E-\varepsilon, E+\varepsilon) > 0$ for all $\varepsilon > 0\}$.

The boundary behavior of m is linked to the Radon-Nikodym derivative $D\mu$ of μ, which in turn determines the decomposition of μ, see e.g. [45].

Theorem 8.1.3. *Let μ be a finite Borel measure and m its Borel transform. Then the limit*

$$\operatorname{Im}\big(m(E)\big) = \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(m(E+i\varepsilon)\big) \tag{8.1.4}$$

exists a.e. with respect to both μ and Lebesgue measure (finite or infinite) and

$$D\mu(E) = \frac{1}{\pi} \operatorname{Im}\big(m(E)\big) \tag{8.1.5}$$

whenever $D\mu(E)$ *exists. Moreover, the set* $\{E|\mathrm{Im}\big(m(E)\big) = \infty\}$ *is a support for the singular continuous part and* $\{E|\mathrm{Im}\big(m(E)\big) < \infty\}$ *is a minimal support for the absolutely continuous part.*

Fiber properties of μ can also be characterized through m. In the rest of this subsection, we briefly review the power-law extension of the Gilbert-Pearson subordinacy theory [15,16], developed in [26].

For simplicity, consider the right half line operator (2.3.1) on $\ell^2(\mathbb{Z}^+)$ with boundary condition $u(1) = \cos\varphi, u(0) = \sin\varphi$ for some $\varphi \in (-\pi/2, \pi/2]$. Let μ be the spectral measure. In this case, the Borel transform of μ is also called the Weyl-Titchmarsh m-function.

For any function $u : \mathbb{Z}^+ \to \mathbb{C}$ and $\ell \in \mathbb{R}^+$, define

$$\|u\|_\ell := \Big[\sum_{n=1}^{[\ell]} |u(n)|^2 + (\ell - [\ell])|u([\ell]+1)|^2\Big]^{1/2}. \tag{8.1.6}$$

Suppose u and v solve $Hu = Eu$ with orthogonal boundary conditions $\begin{pmatrix} u(1) & v(1) \\ u(0) & v(0) \end{pmatrix} = R_\varphi$, a matrix of rotation by φ. Now given any $\varepsilon > 0$, we define a length $\ell(\varepsilon) \in (0, \infty)$ by requiring the equality

$$\|u\|_{\ell(\varepsilon)} \cdot \|v\|_{\ell(\varepsilon)} = \frac{1}{2\varepsilon}. \tag{8.1.7}$$

The function $\ell(\varepsilon)$ is a well defined monotonely decreasing continuous function which goes to infinity as ε goes to 0, and we also have $\frac{1}{2\varepsilon} \geqslant \frac{1}{2}([\ell]-1)$. It turns out that the boundary behavior of $m(E+i\varepsilon)$ is lined in a quantitative way to $\frac{\|u\|_{\ell(\varepsilon)}}{\|v\|_{\ell(\varepsilon)}}$, thus to the power-law behavior of solutions.

Lemma 8.1.8 (J.-Last inequality, [26]). *For* $E \in \mathbb{R}$ *and* $\varepsilon > 0$,

$$\frac{5-\sqrt{24}}{|m(E+i\varepsilon)|} < \frac{\|u\|_\ell}{\|v\|_\ell} < \frac{5+\sqrt{24}}{|m(E+i\varepsilon)|}. \tag{8.1.9}$$

From Lemma 8.1.8, one can easily recover the original results of Gilbert-Pearson [16] with a simpler proof, on top of a strengthening of their theory. The above inequality links the power-law behavior of the generalized eigen-functions of $Hu = Eu$ and the boundary behavior of the Borel transform of the spectral measure μ in a quantitative way. A particular consequence of Lemma 8.1.8 is

Lemma 8.1.10. *For any* $E \in \mathbb{R}$ *and* $0 < \gamma < 1$, *suppose there is a sequence of positive numbers* $\varepsilon_k \to 0$ *and an absolute constant* $C > 0$ *so that both* u, v *satisfy*

$$C^{-1}\ell_k^\gamma \leqslant \|u\|_{\ell_k}^2 \leqslant C\ell_k^{2-\gamma} \tag{8.1.11}$$

where $\ell_k = \ell(\varepsilon_k)$ *is given by* (8.1.7). *Then*

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{1-\gamma}|m(E+i\varepsilon)| < \infty. \tag{8.1.12}$$

8.2. Spectral continuity Fix $0 < \gamma < 1$. If (8.1.12) holds for μ a.e. E, we say measure μ is (upper) γ-spectral continuous. Define the (upper) spectral dimension of

μ to be

$$s(\mu) = \sup\{\gamma \in (0,1) : \ \mu \text{ is } \gamma\text{-spectral continuous}\}. \tag{8.2.1}$$

In this part, we focus on the quantitative spectral continuity and the lower bound of the spectral dimension. Our spectral continuity result does not neccessarily require quasiperiodic structure of the potential and can be generalized to wider contexts (so-called β-almost periodic potential, see [32], - a class, that includes, for example, some skew shift potentials). The general result of [32] only goes in one direction. However, in the important context of analytic quasiperiodic operators this leads to a sharp if-and-only-if result. Let H be defined as in (2.3.2) with quasiperiodic potential:

$$(Hu)(n) = u(n+1) + u(n-1) + \nu(\theta + n\alpha)u(n), \ \ \theta, \alpha \in \mathbb{T}, \ \ \nu : \mathbb{T} \mapsto \mathbb{R}. \tag{8.2.2}$$

Theorem 8.2.3 ([32])**.** *Let H be as in (8.2.2) with real analytic potential ν and μ be the spectral measure*[1]*. Assume $L(E) > 0$ for all $E \in \mathbb{R}$. For any $\theta \in \mathbb{T}$, $s(\mu) = 1$ if and only if $\beta(\alpha) = \infty$.*

Remark 8.2.4. The theorem also holds locally for any spectral projection onto the subset where the Lyapunov exponent is positive.

Remark 8.2.5. The 'if' part will be a consequence of Theorem 8.2.7 which can be viewed as a quantitative strengthening of the results of Gordon type (Lemma 2.8.4). The 'only if' part follows from the general analytic Theorem 8.3.1 which can be viewed as a weakening/extention of localization type results for large β.

Note that spectral continuity captures the $\liminf$ power-law behavior of $m(E + i\varepsilon)$, while the corresponding $\limsup$ behavior is linked to the Hausdorff dimension [13]. One can easily check that $\dim_H(\mu) \leqslant s(\mu) \leqslant \dim_P(\mu)$, where $\dim_H(\mu)/\dim_P(\mu)$ denote the Hausdorff/packing dimension of a measure in the usual sense.

Theorem 8.2.6 (Simon, [44])**.** *Suppose H is an ergodic Schrödinger operator as in (2.3.2) with positive Lyapunov exponent. For a.e. phase ω, $\dim_H(\mu) = 0$.*

Let H be as in (8.2.2) and μ be the spectral measure. We have the following quantitative lower bound of the spectral dimension.

Theorem 8.2.7. *Suppose ν is Lipschitz continuous. Let*

$$\Lambda := \sup_{E \in \sigma(H), n, \theta} \frac{1}{n} \ln \|A_n(\theta)\|. \tag{8.2.8}$$

There exists absolute constant $C > 0$ such that for any $\theta \in \mathbb{T}$,

$$s(\mu) \geqslant 1 - \frac{C\Lambda}{\beta(\alpha)}.$$

[1] The discrecte Schrödinger operator has multiplicity two. In many cases, it is enough to consider the so called maximal spectral measure given by $\mu = \mu_{\delta_0} + \mu_{\delta_1}$, where μ_{δ_0} and μ_{δ_1} are defined as in (2.1.1).

The general version of Theorem 8.2.7 is actually more robust and only requires some regularity of ν, which allows us to obtain new results for other popular models, such as the critical almost Mathieu operator, Sturmian potentials, and others. Lower bounds on spectral dimension also have immediate applications to the lower bounds on packing/box counting dimensions and on quantum dynamics(upper transport exponents). The method developed in [32] for bounded $SL(2,\mathbb{R})$ case can also be generalized to study the unbounded case (e.g. the Maryland model) and the non-Schrödinger case (e.g. the Extended Harper's model) [21,46].

For simplicity, we only prove the right half line case and we also assume (5.0.1) holds. According to Lemma 8.1.10, to prove spectral continuity, it is enough to obtain power-law estimate (8.1.11) for half-line solution u of $Hu = Eu$ with any boundary condition φ.

First, for β large, the system can be approximated by a periodic one exponentially fast in the following sense.

Lemma 8.2.9. *Let q_n be given as in (5.0.1). For any $\beta < \beta(\alpha)$, any $\theta \in \mathbb{T}$, we have*

$$\|A_{q_n}(\theta) - A_{q_n}(\theta + q_n\alpha)\| \leqslant e^{(-\beta+2\Lambda)q_n}. \tag{8.2.10}$$

The ultimate goal is to estimate $\|A_{Nq_n}\|$ by the size of q_n for $N \sim e^{c\beta q_n}$. This eventually leads to the desired power-law for u by (2.6.3). We will conclude this in the end of this part. The standard rational approximation fails here since the error terms may reach the size of $e^{N\Lambda} \sim e^{e^{c\beta q_n}}$. We need some quantitative telescoping argument.

Lemma 8.2.11. *Suppose G is a two by two matrix satisfying*

$$\|G^j\| \leqslant M < \infty, \quad \text{for all } 0 < j \leqslant N \in \mathbb{N}^+, \tag{8.2.12}$$

where $M \geqslant 1$ only depends on N. Let $G_j = G + \Delta_j$, $j = 1, \cdots, N$, be a sequence of two by two matrices with

$$\delta = \max_{1 \leqslant j \leqslant N} \|\Delta_j\|. \tag{8.2.13}$$

If

$$NM\delta < 1/2, \tag{8.2.14}$$

then for any $1 \leqslant n \leqslant N$

$$\|\coprod_{j=1}^{n} G_j - G^n\| \leqslant 2NM^2\delta. \tag{8.2.15}$$

Combining (8.2.10) with this lemma, one can show that A_{Nq_n} is close to $A_{q_n}^N$ up to the size of $\|A_{q_n}^N\|$. Now the question is reduced from the dynamical behavior of A_{Nq_n} to the algebraic properties of $A_{q_n}^N$. We need some additional linear algebraic facts about $SL(2,\mathbb{R})$ matricies.

Lemma 8.2.16. *Suppose* $G \in SL(2,\mathbb{R})$ *with* $2 < |\mathrm{Tr}\, G| \leqslant 6$*. There exists an invertible matrix* B *such that*

$$G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1} \tag{8.2.17}$$

where $\rho^{\pm 1}$ *are the two conjugate real eigenvalues of* G, $|\det B| = 1$ *and*

$$\|B\| = \|B^{-1}\| < \frac{\sqrt{\|G\|}}{\sqrt{|\mathrm{Tr}\, G| - 2}}. \tag{8.2.18}$$

If $|\mathrm{Tr}\, G| > 6$*, then* $\|B\| \leqslant \frac{2\sqrt{\|G\|}}{\sqrt{|\mathrm{Tr}\, G| - 2}}$.

Lemma 8.2.19. *Suppose* $G \in SL(2,\mathbb{R})$ *has eigenvalues* $\rho^{\pm 1}$, $\rho > 1$*. For any* $k \in \mathbb{N}$*, if* $\mathrm{Tr}\, G \neq 2$*, then*

$$G^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \Big(G - \frac{\mathrm{Tr} G}{2} \cdot I\Big) + \frac{\rho^k + \rho^{-k}}{2} \cdot I. \tag{8.2.20}$$

Otherwise, $G^k = k(G - I) + I$.

Assume further that $\big||\mathrm{Tr}\, G| - 2\big| < \tau < 1$*. Then there are universal constants* $1 < C_1 < \infty, c_1 > 1/3$ *such that for* $1 \leqslant k \leqslant \tau^{-1}$*, we have*

$$c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1\,, \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k. \tag{8.2.21}$$

By Lemma 8.2.16, when the trace of A_{q_n} is away from 2, we have the following decomposition

$$A_{q_n} = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}, \quad \|B\| = \|B^{-1}\| \leqslant \frac{2\sqrt{\|A_{q_n}\|}}{\sqrt{|\mathrm{Tr}\, A_{q_n}| - 2}} \tag{8.2.22}$$

and the matrix product turns into a scalar product,

$$A_{q_n}^N = B \begin{pmatrix} \rho^N & 0 \\ 0 & \rho^{-N} \end{pmatrix} B^{-1}, \quad \|A_{q_n}^N\| \leqslant \|B\|^2 |\rho|^N. \tag{8.2.23}$$

By Lemma 8.2.19, when the trace of A_{q_n} is close to 2, $A_{q_n}^N$ behaves almost linearly in N:

$$A_{q_n}^N \sim N\big(A_{q_n} - \frac{1}{2}\mathrm{Tr}\, A_{q_n}\big) + I. \tag{8.2.24}$$

The above two asymptotic behaviors of $A_{q_n}^N$ allow us to study the spectral measure of the following two sets:

$$S_1 = \limsup_{n\to\infty} \{E : \ |\mathrm{Tr} A_{q_n}| > 2 + e^{-10\Lambda q_n}\}, \tag{8.2.25}$$

$$S_2 = \limsup_{n\to\infty} \{E : \big||\mathrm{Tr} A_{q_n}| - 2\big| < e^{-10\Lambda q_n}\}. \tag{8.2.26}$$

To estimate the spectral measure of S_1, we use the idea of a Gordon-type argument to estimate the lower bound of the solution. Recall the key step to prove

Lemma 2.8.4 is that for $G \in SL(2,\mathbb{R})$ and $X \in \mathbb{C}^2$,

$$\max\{\|GX\|, \|G^{-1}X\|\} \geqslant \frac{1}{2}|\mathrm{Tr}G| \cdot \|X\|. \tag{8.2.27}$$

If $E \in S_1$, rougly speaking, we have a sequence of scales q_n such that the trace of A_{q_n} is large. Putting (8.2.10),(8.2.15),(8.2.22) and (8.2.27) together, we can show that there are integer sequences $x_{q_n} \to \infty$ *independent of* E, such that

$$|u(x_{q_n})| > e^{q_n}, \tag{8.2.28}$$

where u solves the half-line problem $Hu = Eu$ with any boundary condition. The following extended Schnol's Theorem shows that such E must have spectral measure zero.

Lemma 8.2.29 (Extended Schnol's Theorem, [32])**.** *Fix any* $y > 1/2$. *For any sequence* $|x_k| \to \infty$*(where the sequence is independent of* E*), for spectrally a.e.* E*, there is a generalized eigenvector* u *of* $Hu = Eu$*, such that*

$$|u(x_k)| < C(1+|k|)^y.$$

For S_2, note that $A_q(E)$ is a polynomial in E with degree at most q. If the trace is close to 2, the following preimage estimate of a polynomial reduces the set in S_2 to several small intervals of width at most $e^{-5\Lambda q_n}$.

Lemma 8.2.30 ([29])**.** *Let* $p \in \mathcal{P}_{n;n}(\mathbb{R})$ *with* $y_1 < \cdots < y_{n-1}$ *the local extrema of* p. *Let*

$$\zeta(p) := \min_{1 \leqslant j \leqslant n-1} |p(y_j)| \tag{8.2.31}$$

and $0 \leqslant a < b$. *Then,*

$$|p^{-1}(a,b)| \leqslant 2\mathit{diam}(z(p-a)) \max\left\{\frac{b-a}{\zeta(p)+a}, \left(\frac{b-a}{\zeta(p)+a}\right)^{\frac{1}{2}}\right\} \tag{8.2.32}$$

where $z(p)$ *is the zero set of* p *and* $|\cdot|$ *denotes the Lebesgue measure.*

The definition of m-function gives a simple relation $\mu(E-\varepsilon, E+\varepsilon) \leqslant 2\varepsilon\, \mathrm{Im}M(E+i\varepsilon)$, where the right-hand side can be estimated again by subordinacy theory (Lemma 8.1.8) with the help of (8.2.24). Alltogether, one can show that for β large enough, $\mu(\{E : \big||\mathrm{Tr}A_{q_n}| - 2\big| < e^{-10\Lambda q_n}\}) < e^{-\Lambda q_n}$. Then Borel Cantelli lemma immediately implies $\mu(S_2) = 0$.

In conclusion, we have the following key estimate for the trace of the transfer matrices.

Theorem 8.2.33. *For* $\beta > 40\Lambda$ *and* μ *a.e.* E*, there is* $K(E)$ *such that*

$$|\mathrm{Trace}A_{q_n}(E)| < 2 - e^{-10\Lambda q_n}, \quad n \geqslant K(E). \tag{8.2.34}$$

Combining this trace estimate with previous algebraic facts (8.2.15) and (8.2.24), one has

Lemma 8.2.35. *There is a sequence of positive integers* $N_k \to \infty$ *such that for* $0 < \gamma < 1$, *if*

$$\beta > \frac{100\Lambda}{1-\gamma}, \tag{8.2.36}$$

then

$$\sum_{n=1}^{N_k \cdot q_k} \|A_n(E)\|^2 \leqslant (N_k \cdot q_k)^{2-\gamma}, \quad k \geqslant K(E). \tag{8.2.37}$$

Now (8.1.11) follows from (2.6.3) for any boundary condition φ.

8.3. Arithmetic criteria In this part, we focus on the spectral singularity and the quantitative upper bound of $s(\mu)$. For simplicity, we only state and prove the following upper bound for the right half line AMO. The same result holds for general analytic potentials with positive Lyapunov exponent, which together with Theorem 8.2.7 will compelte the proof of Theorem 8.2.3.

Theorem 8.3.1. *Let* H *be the AMO given as in* (2.7.1). *Assume that* $\lambda > 1$. *There exists* $\varphi \in (-\pi/2, \pi/2]$ *and an absolute constant* c *such that for any* $\theta \in \mathbb{T}$ *if* $\beta(\alpha) < \infty$ *then for the associated half line spectral measure* μ, *we have that*

$$s(\mu) \leqslant \frac{1}{1+c/\beta} < 1. \tag{8.3.2}$$

Lemma 8.3.3. *For any* E *there is a* n_0 *such that for any* $n > n_0$, *there exists an interval* $\Delta_n \subset \mathbb{T}$ *satisfying*

$$\mathrm{Leb}(\Delta_n) \geqslant \frac{1}{8n}, \quad \inf_{\theta \in \Delta_n} \frac{1}{n} \ln \|A_n(\theta)\| > \frac{1}{4} \ln \lambda. \tag{8.3.4}$$

Moreover, for all q_n *large (depending on* n_0*), for any* θ, *and any* $N \in \mathbb{N}$, *there is* $j_N \in [2Nq_n, 2(N+1)q_n)$ *such that*

$$\|A_{j_N}(\theta, E)\| > e^{\frac{1}{36} q_n \ln \lambda}. \tag{8.3.5}$$

Lemma 8.3.6. *For any* $E \in \mathbb{R}$ *and* $\beta = \beta(\alpha) < \infty$, *there is* $\ell_0 = \ell_0(E, \beta)$ *such that for* $\ell > \ell_0$, *and any* $\theta \in \mathbb{T}$, *the following holds:*

$$\sum_{k=1}^{\ell} \|A_k(\theta, E)\|^2 \geqslant \ell^{1+\frac{2c}{\beta}}. \tag{8.3.7}$$

Proof of Theorem 8.3.1: For any φ, we have

$$\|u^\varphi\|_\ell^2 + \|v^\varphi\|_\ell^2 \geqslant \frac{1}{2} \sum_{k=1}^{\ell} \|A_k(\theta)\|^2. \tag{8.3.8}$$

Therefore, (8.3.7) implies that $\|u^\varphi\|_\ell^2 + \|v^\varphi\|_\ell^2 \geqslant \ell^{1+\frac{2c}{\beta}}$ for ℓ large.

On the other hand, Last and Simon showed in [38] that, for μ-a.e. E, there exist φ and $C = C(E) < \infty$, such that for large ℓ,

$$\|u^\varphi\|_\ell \leqslant C\ell^{1/2} \ln \ell. \tag{8.3.9}$$

Combining (8.3.8) and (8.3.9), we have

$$\|v^\varphi\|_\ell \geqslant \ell^{1/2+c/\beta} \tag{8.3.10}$$

provided $\beta < \infty$ and $\ell > \ell_0(E, \beta)$. For any $\varepsilon > 0$ (small), let $\ell = \ell(\varepsilon)$ be given as in (8.1.7). By (8.1.9), one has for any $\gamma \in (0, 1)$,

$$\varepsilon^{1-\gamma}|m_\varphi(E+i\varepsilon)| \geqslant \frac{1}{\left(2\|u^\varphi\|_\ell \|v^\varphi\|_\ell\right)^{1-\gamma}} \cdot (5-\sqrt{24}) \frac{\|v^\varphi\|_\ell}{\|u^\varphi\|_\ell} \geqslant c_\gamma \ell^{(1+c/\beta)\gamma - 1} \cdot \ln^{-2} \ell$$

where $c_\gamma > 0$ only depends on γ. Now let $\gamma_0 = \frac{1}{1+c/\beta} < 1$. We have for any $\gamma > \gamma_0$,

$$\varepsilon^{1-\gamma}|m_\varphi(E+i\varepsilon)| \geqslant c_\gamma \ell^{\gamma/\gamma_0 - 1} \cdot \ln^{-2} \ell \to \infty$$

as $\varepsilon \to 0$. Therefore, $s(\mu) \leqslant \gamma_0$, according to the definition (8.2.1). □

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