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Almost Commuting Elements in Non-Commutative Symmetric Operator Spaces

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Dedication

To God, my family, Timur Oikhberg, and Svetlana Jitomirskaya

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Curriculum Vitae

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Abstract of the Dissertation

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I. Almost Commuting Operators: John von Neumann, in his formulation of the uncertainty principle [24] was the first to consider the classic problem which asks whether two "almost commuting" operators are small perturbations of commuting operators. Von Neumann, however, did not mention any norm. The first appearance of the problem in the literature is given by Rosenthal [22] and Halmos [11]. Although Rosenthal considers the problem with respect to the Hilbert-Schmidt norm, historically most of the results are with respect to the operator norm. In fact, the first positive result was proved by H. Lin in [16] with respect to the operator norm. In 2009 Lev Glebsky in [10] proved an analog of Lin's theorem with respect to the *normalized* Hilbert-Schmidt norm. Glebsky's result was refined in [9] by Filonov and Kachkovskiy. Motivated by these results, we consider a natural generalization of the problem. In particular, we look at the problem with respect to the *normalized* Schatten p-norms for $1 \leq p < \infty$ and have established several theorems that are analogs of Lin and Glebsky's work. Moreover, for $p \neq 2$ the corresponding Schatten space

is not a Hilbert space and our results use general Banach space techniques along with some tools from classical and harmonic analysis. For $p = 2$ we recover Filonov and Kachkovskiy's result with the same $\delta(\epsilon)$ relationship. We have also extended our results to finite von Neumann algebras equipped with an (n.s.f) trace which will be defined later.

II. Normal Completions: In this portion of the dissertation we take a detour and consider another matrix problem. In [1] Bhatia and Choi ask the following question: What pairs of matrices, (B, C) can be the off-diagonal entries of $2n \times 2n$ normal matrices of the form

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D \in M_n(\mathbb{C})$? We provide some explicit "normal completions" for different pairs of matrices (B, C) which do not appear in the literature.

Introduction:

We investigate a variant of an old problem in linear algebra and operator theory that was first considered by John von Neumann [24] and later popularized by Peter Rosenthal [22] and Paul Halmos [11]: **Must almost commuting matrices be near commuting matrices?** To be more precise, we consider the following classic problem:

Problem 1: Suppose that $A, B \in M_n(\mathbb{C})$. Given $\epsilon > 0$, is there a $\delta > 0$ such that for each $n \in \mathbb{N}$ with $||A||, ||B|| \leq 1$ and $||AB - BA|| < \delta$, then does there exist $X, Y \in M_n(\mathbb{C})$ with $XY = YX$ and $||A - X|| + ||B - Y|| < \epsilon$?

Here, $||.||$ denotes a norm on $M_n(\mathbb{C})$. We also note that it is important that $\delta = \delta(\epsilon)$ only depends on ϵ and not n, the dimension of our Hilbert space $M_n(\mathbb{C})$. Surprisingly, the answer to **Problem 1** is negative in general, but positive for some special cases. We now present a brief overview of the known results and defer the relevant notation and definitions to Chapter 1 and Chapter 2.

0.1 Historical Results

The first results on our classic problem were dimension dependent. To be more specific, the initial progress on the problem had δ in the statement of the theorem depend not only on ϵ but on the dimension of the matrices. These results are not hard to obtain; many of the proofs use a simple compactness argument. The next wave of progress consisted of counter-examples. In the 1980's a series of "almost commuting" pairs of matrices which are "far" from any commuting approximates have been obtained. M-D Choi, for example, in [4] proved that there exist $A, B \in M_n(\mathbb{C})$ both contractions, with A a self-adjoint, and $||A||, ||B|| \leq 1$, such that:

$$||[A,B]|| < 2/n, \text{ but } \inf_{\{(X,Y):XY=YX\}}\{||A-X|| + ||B-Y||\} \geq 1 - 1/n.$$

Hence, for $\epsilon < 1/2$ there does not exist a $\delta > 0$ satisfying the hypothesis and conclusion of **Problem 1**. In the same spirit, in 1989, Ruy Exel and Terry Loring in [8] gave an example of a pair of of "almost commuting" unitaries that are bounded away from any commuting approximates. In particular, they used a family of unitary matrices, often referred to as "Voiculescu's Unitaries," $U_n, V_n \in M_n(\mathbb{C})$ and showed that there exists a universal constant $C > 0$ such that:

$$||[U_n, V_n]|| \to 0, n \to \infty \text{ yet } ||U_n - X|| + ||V_n - Y|| > C > 0.$$

for all commuting matrices, $XY = YX$. The norm under consideration here is the usual operator norm. Earlier partial results were obtained by Voiculescu [23] and Davidson [6]. In Chapter 7 we will take a close look at Exel and Loring's counter-example and utilize their argument, with modifications, to extend their result on the operator norm to the Schatten p-norms for $1 < p \leq \infty$.

These results for **Problem 1** do not seem encouraging. They seem to suggest that the statement is false for most classes of matrices in $M_n(\mathbb{C})$. However, in 1997 H. Lin [16] proved that **Problem 1** is true for almost commuting self-adjoint contractions A and B. Lin proved that there exist a function $f : (0, \infty) \to (0, \infty)$ with $\lim_{x \to 0+} f(x) = 0$, such that

$$||A - X|| + ||B - Y|| \leq f(||AB - BA||) \text{ for all } X, Y \text{ with } XY = YX.$$

Lin, however, did not provide any estimates on f, but M. Hastings in [12] attempted to provide a constructive proof of Lin's theorem with explicit estimates on the function f. In particular, he tried to show that there existence of f with $\lim_{x\to 0+} f(x)/x^{\alpha} = 0$ for any $\alpha < 1/6$. Recently, however, a error was found in Hastings proof. In a private communication with Ilya Kachkovskiy the author has learned that Kachkovskiy and Safarov have established a constructive version of Lin's theorem with optimal exponent of 1/2.

Loring and Hastings conjectured [13] that, for several classes $\mathcal{C} \subset M_n$ of "sufficiently symmetric" matrices (arising from mathematical physics) for any two contractions $A, B \in \mathcal{C}$, with $|||[A, B]||| < \epsilon$ there exist commuting contractions, $X, Y \in \mathcal{C}$ such that $||A - X|| + ||B - Y|| < f(|||[A, B]|||)$.

In a series of papers Terry Loring and Adam Sorensen have extended Lin's results to classes of matrices other than self-adjoint ones. In [17], [18], and [19], Loring and Sorensen show that almost commuting self-adjoint symmetric matrices are close to a pair of commuting self-adjoint, symmetric matrices. They also prove that the same holds if symmetric is replaced by self-dual and real orthogonal.

Although, historically, most of the research on the problem is done with respect to the operator norm, Rosenthal's original formulation of the problem in the literature [22] is with respect to the Hilbert-Schmidt norm. In 2009, Lev Glebsky [10] proved an analog of H. Lin's theorem with the operator norm replaced by the *normalized* Hilbert-Schmidt norm.

Motivated by these results we examine **Problem 1** for different classes of matrices in $M_n(\mathbb{C})$ and consider the "almost commuting implies near commuting" problem for different norms $|||.|||$. In particular we study Problem I with respect to the *normalized* Schatten p-norms, which is defined for $A \in M_n(\mathbb{C})$ as $||A||_p = (1/n \sum_{j=1}^{n} s_j(A)^p)^{1/p}$. Here $1 \leq p < \infty$

and $s_j(A)$ denotes the jth singular value of the matrix A. Recall that the singular values of an operator A are just the eigenvalues of $\sqrt{A^*A}$. Moreover, we have generalized our results to the setting of von Neumann algebras equipped with a trace and our results for the *normalized* Schatten p-norms, $1 \leq p < \infty$, arise as corollaries.

0.2 A Generalization of Problem 1

We may consider a more general formulation of **Problem 1**. In particular, we would like to ask the following: Let $\epsilon > 0$. If $q(X_1, ..., X_k)$ is a polynomial in several non-commutative variables $X_1, ..., X_k$ in some Banach algebra $(\mathcal{X}, ||.||)$, does there exist $\delta > 0$ such that if $X_j \in \mathcal{X}$ with $||X_j|| \leq 1$ for $1 \leq j \leq k$ and $||q(X_1, ..., X_k)|| < \delta$, then are there $Y_\ell \in \mathcal{X}$ such that $q(Y_1, ..., Y_k) = 0$ and $\sum_j ||X_j - Y_j|| < \epsilon$?

Remark 1. *Note that when* $q(X, Y) = XY - YX$ *and* $\mathcal{X} = M_n(\mathbb{C})$ *equipped with the standard operator norm, our formulation is equivalent to the statement of* ***Problem 1****. In [13] Hastings and Loring consider polynomials other than* $q(X, Y) = XY - YX$ *and determine whether almost solutions of these non-commutative polynomials are perturbations of exact solutions. For example, they consider whether near solutions of the matrix equation* $X^2 + Y^2 + Z^2 = I$ *with* $[X, Y] = [X, Z] = [Y, Z] = 0$ *are close to exact solutions.*

In this framework, we have been able to establish several results which are analogues of Lin's theorem with the added advantage that our proofs are constructive and provide an explicit $\delta = \delta(\epsilon)$ relationship. We have also recovered N. Filonov and I. Kachkovskiy results [9] as special cases of a more general theorem with the same estimates. Note that N. Filonov and I. Kachkovskiy proved the same theorem as L. Glebsky in [10] with better estimates. Although our work began with finite-dimensional operators, matrices, we have extended most of our results to von Neumann algebras $\mathcal{M}$ that have a (n.s.f) trace τ and we state our theorems in this setting. We will now formulate our main results and defer the notation and preliminary definitions we will use to Chapter 1 and Chapter 2.

0.3 Main Results

Theorem 0.3.1. *Consider the non-commutative polynomial $q(x,y) = xy - yx$. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with an (n.s.f) trace τ. Let $||.||$ denote the usual operator norm and suppose $(\mathcal{E}, \|\!|.|\!\|)$ is a rearrangement invariant Banach function space on $[0, \tau(1))$, satisfying the lower p-estimate for some p and constant 1, where $1 \leq p < \infty$. Assume that $a \in \mathcal{M}_{sa}$ is a self-adjoint operator such that $||a|| \leq 1$, and $b \in \mathcal{E}(\mathcal{M}, \tau)$ is an operator that satisfies $\|\!|b|\!\| \leq 1$. Then there exists operators $\tilde{a}$, with $||\tilde{a}|| \leq ||a||$, and an operator $\tilde{b}$, commuting with $\tilde{a}$, so that $||a - \tilde{a}|| + \|\!|b - \tilde{b}|\!\| \leq K \|\!|q(a,b)|\!\|^{\frac{1}{p+2}}$. Here, the constant K does not depend on our von Neumann algebra. Moreover, if b is self-adjoint, then $\tilde{b}$ can also be chosen to be self-adjoint as well.*

Corollary 0.3.1. *Consider the non-commutative polynomial $q(X,Y) = XY - YX$. Let $||.||$ denote the usual operator norm and let $||.||_p$ denote the normalized Schatten p-norm where $1 \leq p < \infty$, and let H_n be the class of hermitian matrices in $M_n(\mathbb{C})$. Then for any matrix $A \in H_n$ such that $||A|| \leq 1$, and $B \in H_n$ with $||B||_p \leq 1$ there exists commuting contractions X, Y with $X \in H_n$ such that $||A - X||_p + ||B - Y||_p \leq K ||q(A,B)||_p^{1/(p+2)}$. Here $K = K(p)$ is a constant independent of the dimension of our matrices. Moreover, if $B \in H_n$ then Y can also be chosen to be self-adjoint as well.*

Corollary 0.3.2. *Suppose that $A \in M_n(\mathbb{C})$ satisfies $||A|| \leq 1$ where $||.||$ is the usual operator norm. Let $||.||_p$ denote the normalized Schatten p-norm $1 \leq p < \infty$. Then there exists a normal matrix N such that $||N - A||_p \leq K||AA^* - A^*A||_p^{1/(p+2)}$, where $K = K(p)$ is a universal constant independent of the dimension of the matrix A.*

We prove the theorems stated above in Chapter 3. We also prove that almost "anti-commuting" self adjoint matrices are "nearly anti-commuting." In particular, we prove the following result in Chapter 4:

Theorem 0.3.2. *Consider the non-commutative polynomial $q(x, y) = xy+yx$. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with an (n.s.f) trace τ. Let $||.||$ denote the usual operator norm and suppose $(\mathcal{E}, \|\!|.\|\!|)$ is a rearrangement invariant Banach function space on $[0, \tau(1))$, satisfying the lower p-estimate for some p and constant 1, where $1 \leq p < \infty$. Assume that $a \in \mathcal{M}_{sa}$ is a self-adjoint operator such that $||a|| \leq 1$, and $b \in \mathcal{E}(\mathcal{M}, \tau)$ is an operator that satisfies $\|\!|b\|\!| \leq 1$. Then there exists operators $\tilde{a}$, with $||\tilde{a}|| \leq ||a||$, and an operator $\tilde{b}$, anti-commuting with $\tilde{a}$, so that $||a - \tilde{a}|| + \|\!|b - \tilde{b}\|\!| \leq K_1 \|\!|q(a, b)\|\!|^{\frac{1}{p+2}}$. Here, the constant $K_1 = K_1(p)$ does not depend on our von Neumann Algebra. Moreover, if b is self-adjoint, then $\tilde{b}$ can also be chosen to be self-adjoint.*

Corollary 0.3.3. *Consider the non-commutative polynomial $q(X, Y) = XY + YX$. Let $||.||$ denote the usual operator norm and let $||.||_p$ denote the normalized Schatten p-norm where $1 \leq p < \infty$. Assume that $A \in M_N(\mathbb{C})$ is a self-adjoint matrix such that $||A|| \leq 1$, and $B \in M_N(\mathbb{C})$ satisfies $||B||_p \leq 1$. Then there exists anti-commuting matrices $\tilde{A}, \tilde{B} \in M_N(\mathbb{C})$ such that $||A - \tilde{A}||_p + ||B - \tilde{B}||_p \leq K_1||AB + BA||_p^{1/(p+2)}$ where $K_1 = K_1(p)$ is a constant that does not depend on the dimension of our matrices. Moreover, $\tilde{A}$ is self-adjoint and if B*

is self-adjoint, then $\tilde{B}$ can also be chosen to be self-adjoint.

Next, we look at the operator equation $AB = \omega BA$ and its approximate version. We consider the case when $\omega \in \mathbb{T}$ and provide the motivation for this restriction. In this setting, we have been able to establish the following theorem in Chapter 5:

Theorem 0.3.3. *Consider the non-commutative polynomial $q(x, y) = xy - \omega yx$ where ω is a root of unity. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with an (n.s.f) trace τ. Let $||.||$ denote the usual operator norm and suppose $(\mathcal{E}, \|\!|.\|\!|)$ is a rearrangement invariant function space on $[0, \tau(1))$, satisfying the lower p-estimate for some p and constant 1, where $1 \leq p < \infty$. Assume that $a \in \mathcal{M}$ is a unitary operator, and $b \in \mathcal{E}(\mathcal{M}, \tau)$ satisfies $\|\!|b\|\!| \leq 1$. Then there exist operators $\tilde{a}, \tilde{b}$ such that $q(\tilde{a}, \tilde{b}) = 0$ and $||a - \tilde{a}|| + \|\!|b - \tilde{b}\|\!| \leq K_2 \|\!|q(a, b)\|\!|^{1/(p+2)}$. Here $K_2 = K_2(p, \omega)$ is a constant independent of the von Neumann algebra.*

Corollary 0.3.4. *Consider the non-commutative polynomial $q(X, Y) = XY - \omega YX$ where ω is a root of unity. Let $||.||$ denote the usual operator norm and let $||.||_p$ denote the normalized Schatten p-norm where $1 \leq p < \infty$. Assume that $A \in M_N(\mathbb{C})$ is a unitary matrix and $B \in M_N(\mathbb{C})$ satisfies $||B||_p \leq 1$. Then there matrices $\tilde{A}, \tilde{B} \in M_N(\mathbb{C})$ such that $\tilde{A}\tilde{B} = \omega \tilde{B}\tilde{A}$ and $||A - \tilde{A}||_p + ||B - \tilde{B}||_p \leq K_2 ||AB - \omega BA||_p^{1/(p+2)}$ where $K_2 = K_2(p, \omega)$ is a constant that does not depend on the dimension of our matrices.*

We also look at the same problem with the added constraint that the operators a, b be self-adjoint. Formally, we consider:

Equation 1. *$ab = \omega ba$, with the restriction that $a = a^*$ and $b = b^*$.*

As we will see, Equation 1 is quite restrictive as illustrated by the followig Proposition which is proved in Chapter 6.

Proposition 0.3.1. *If $\omega \in \mathbb{C} - \{-1, +1\}$, and $a, b \in B(\mathcal{H})$ satisfy* **Equation 1***, then $ab = ba = 0$.*

Hence, following the theme of our text, we look at the approximate version of Equation 1. That is, we consider the case when $a = a^*$, $b = b^*$ and $\|ab - \omega ba\|$ is "small" and show that a and b can be approximated by $\tilde{a}$ and $\tilde{b}$, respectively, so that $||a - \tilde{a}||$ and $\|b - \tilde{b}\|$ are "small," and $\tilde{a}\tilde{b} = \tilde{b}\tilde{a} = 0$. In this situation, we have been able to establish the following theorem in Chapter 6:

Theorem 0.3.4. *Suppose $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra and let $a \in \mathcal{M}_{sa}$ and assume that $||.||$ denotes the operator norm on $\mathcal{M}$ and let $b \in \mathcal{E}(\mathcal{M}, \tau)$, where $\|.\|$ is the norm inherited from $\mathcal{E}(\mathcal{M}, \tau)$. Now, let $\omega \in (-1, 1)$ and set $\kappa = \|ab - \omega ba\|$. Then, for any $\epsilon > 0$ the operator a has a spectral projection Q so that*

$$(1) \qquad \begin{aligned} &(i) \quad ||Qa|| \leq \epsilon \\ &(ii) \quad \|b - QbQ\| \leq K(\omega)\epsilon^{-1}\kappa, \end{aligned}$$

Here $K(\omega)$ is a constant that depends only on ω.

Remark 2. *Once this is established, we have that the self-adjoint pair of operators (a, b) is close to a pair of operators $(\tilde{a}, \tilde{b})$ that "trivially" commute up to a factor in the sense that if we take $\tilde{a} = Q^{\perp}a$ and $\tilde{b} = QbQ$, we have $||a - \tilde{a}|| \leq \epsilon$, $\|b - \tilde{b}\| \leq K(\omega)\epsilon^{-1}\kappa$, and $\tilde{a}\tilde{b} = \tilde{b}\tilde{a} = 0$. Hence, in particular, $\tilde{a}\tilde{b} = \omega\tilde{b}\tilde{a}$.*

As a result, we get the following corollary,

Corollary 0.3.5. *Suppose $a = a^*$, $b = b^*$, and $\omega \in (-1, 1)$. Then for every $\epsilon > 0$ there* *exists a* $\delta > 0$ *so that, whenever* $||a|| \leq 1$, *and* $\|ab - \omega ba\| < \delta$, *there exists* b' *and* a' *so that* $\|b - b'\| < \epsilon$ *and* $||a - a'|| < \epsilon$. *Moreover,* $a'b' = b'a' = 0$.

Next we turn to examples of almost solutions to non-commutative polynomials q for which near solutions to the equation $q(x, y) = 0$ cannot be approximated by the exact solutions. A sample result is **Theorem 7.3.1**. In this direction we have been able to establish two counter-examples that do not appear in the literature.

Finally, in Chapter 8, we turn to the problem of finding normal completions of corners of matrices. Perhaps the most interesting result we have established is the following Theorem:

Theorem 0.3.5. *If* $B = diag(B_1, B_2)$ *and* $C = diag(C_1, C_2)$ *are* $2n \times 2n$ *block-diagonal self-adjoint matrices with real entries that satisfy the following system of matrix equations:*

1. $B_1^2 + B_2^2 = C_1^2 + C_2^2$
2. $B_1C_1 + B_2C_2 = C_1B_1 + C_2B_2$

then matrix pair (B, C) *admits a* $4n \times 4n$ *normal completion.*

Chapter 1

von Neumann Algebras

1.1 Topologies on $\mathcal{B}(\mathcal{H})$

Throughout, we will let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the set of all linear operators from a Hilbert space $\mathcal{H}$ to $\mathcal{K}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is **bounded** if there exists a constant M such that $||Tx||_{\mathcal{K}} \leq M||x||_{\mathcal{H}}$ for all $x \in \mathcal{H}$. We denote $||T||$ to be the least constant M so that the inequality above is satisfied. Let $\mathcal{B}(\mathcal{H})$ denote the set of all **bounded** linear operators from $\mathcal{H}$ to itself.

We can equip $\mathcal{B}(\mathcal{H})$ with several different topologies:

- If $||T_n - T|| \to 0$ as $n \to \infty$ where $||.||$ is the operator norm, then we say that $T_n \to T$ in the **norm** or **uniform topology**.
- The topology on $\mathcal{B}(\mathcal{H})$ of pointwise convergence on $\mathcal{H}$ is called the **strong operator topology**. A family of operators, $\{T_n \in \mathcal{B}(\mathcal{H})\}$ converges strongly to an operator T if

$$||T_n(x) - T(x)|| \to 0 \text{ as } n \to \infty \text{ for all } x \in \mathcal{H}$$

- If $F(T_n(x)) \to F(T(x))$ for all linear functionals F on $\mathcal{H}$, we say that $T_n \to T$ in the **weak operator topology** on $\mathcal{B}(\mathcal{H})$

Definition 1.1.1. *A von Neumann Algebra $\mathcal{M}$ on $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ with $\mathbf{1} \in \mathcal{M}$ that is closed under the weak operator topology.*

Remark 3. *Note that for a $*$-algebra being closed in the strong operator topology (SOT) is equivalent to being closed in the weak operator topology (WOT).*

Definition 1.1.2. *The real subspace of $\mathcal{B}(\mathcal{H})$ consisting of all self-adjoint operators is denoted by $\mathcal{B}(\mathcal{H})_{sa}$. The collection of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})_+$. Moreover, we let $\mathcal{M}_+ = \mathcal{M} \cap \mathcal{B}(\mathcal{H})_+$ which is a proper closed generating cone in $\mathcal{B}(\mathcal{H})_{sa}$.*

Von Neumann's Double Commutant theorem shows that the analytic definition is equivalent to a purely algebraic definition as an algebra of symmetries. To be more precise, von Neumann proved the following:

Theorem 1.1.1. *(Double Commutant Theorem) Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$ where $\mathcal{A}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST$ for all $S \in \mathcal{B}(\mathcal{H})$.*

We mention von Neumann's Double Commutant theorem for its historical role as in this dissertation we are mainly going to apply techniques from real and harmonic analysis to establish our results in the von Neumann algebra setting.

1.2 Examples of von Neumann Algebras

1. $\mathcal{B}(\mathcal{H})$ itself, trivially.

2. Let (X, μ) be a finite measure space and consider $\mathcal{M} = L^\infty(X, \mu)$ as a $*$-subalgebra of $\mathcal{B}(L^2(X, \mu))$. Here, $\mathcal{M} = L^\infty(X, \mu)$ acts on $L^2(X, \mu)$ as multiplication operators. For every $f \in L^\infty(X, \mu)$, we can define the operator $M_f : L^2(X, \mu) \to L^2(X, \mu)$ by $M_f(g(x)) = f(x)g(x)$. Then it is easy to see that for each $f \in L^\infty(X, \mu)$, we have that $||M_f|| = ||f||_\infty < \infty$ so that these operators are bounded. The von Neumann algebra generated by these operators is commutative and one can show that all commutative von Neumann algebras arise from such multiplication operators.

3. $M_n(\mathbb{C})$ the space of all $n \times n$ matrices over the complex field.

4. Let Γ be a discrete group and let $l^2(\Gamma)$ be the Hilbert space of all functions $f : \Gamma \to \mathbb{C}$ with $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty$ and inner product $\langle f, g \rangle = \sum_{\gamma \in \Gamma} f(\gamma)\overline{g(\gamma)}$. An orthonormal basis of $l^2(\Gamma)$ is given by the the $\{\epsilon_\gamma\}$ where $\epsilon_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ so that $f = \sum_{\gamma \in \Gamma} f(\gamma)\epsilon_\gamma$ in the l^2 sense. Now for each $\gamma \in \Gamma$ define the unitary operators u_γ by $(u_\gamma f)(\gamma') = f(\gamma^{-1}\gamma')$. Note that $u_\gamma u_\rho = u_{\gamma\rho}$ and that $u_\gamma(\epsilon_\rho) = \epsilon_{\gamma\rho}$. Thus $\gamma \to u_\gamma$ is a unitary group representation called the *left regular representation*. The u_γ are linearly independent so the algebra they generate is isomorphic to the group algebra $\mathbb{C}\Gamma$. The von Neumann Algebra generated by the u_γ is denoted here by $vN(\Gamma)$ and it is known as the "group von Neumann algebra" of Γ.

For a more detailed treatment of von Neumann algebras, the reader is referred to excellent exposition in [2] and [14].

Chapter 2

Non-Commutative Banach Function Spaces

2.1 Introduction

In this section we use the presentation found in [7] to review some parts of the theory of *non-commutative Banach function spaces*, which are spaces of measurable operators associated with a semi-finite von Neumann algebra. These spaces are also known as *non-commutative symmetric operator spaces*. The theory of such spaces emerged as a common generalization of the theory of classical, commutative, rearrangement invariant Banach function spaces and of the theory of symmetrically normed ideals of bounded linear operators in a Hilbert space. These two cases may be considered as the two extremes of the theory; in the first case the underlying von Neumann algebra is the commutative algebra L_∞ on some measure space with integration as the trace; in the second case the underlying von Neumann algebra is $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space with the standard trace. Important special cases of these non-commutative spaces are the non-commutative L_p-spaces, which correspond in the commutative case with the usual L_p-spaces on a measure space, and in the setting of symmetrically normed operator ideals they correspond to the Schatten p-classes $\mathcal{S}_p(\mathcal{H})$.

2.2 von Neumann Algebras equipped with a trace

Let $\mathcal{M}$ be a von Neumann Algebra, and $\mathcal{M}_+$ its positive part.

Definition 2.2.1. *A* **trace** *is a map* $\tau : \mathcal{M}_+ \to [1, \infty)$ *such that:*

1. $\tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$ *for all* $x, y \in \mathcal{M}_+$ *and* $\lambda \in \mathbb{R}_+$
2. $\tau(x^*x) = \tau(xx^*)$ *for all* $x \in \mathcal{M}$

Definition 2.2.2. *A trace is said to be:*

1. **normal** *if* $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ *for any bounded increasing net* $(x_i) \in \mathcal{M}_+$*,*
2. **faithful** *if* $\tau(x) = 0$ *implies that* $x = 0$*,*
3. **finite** *if* $\tau(1) < \infty$*,*
4. **semifinite** *if for any non-zero* $x \in \mathcal{M}_+$*, there exists a non-zero* $y \in \mathcal{M}_+$ *such that* $y \leq x$ *and* $\tau(y) < \infty$*.*

Definition 2.2.3. $\mathcal{M}$ *is called* **semifinite** *if it admits a normal semifinite faithful (n.s.f.) trace.*

Remark 4. *Although the trace is defined initially on* $\mathcal{M}_+$*, we can define the trace on the linear span of all positive elements with finite trace. This just follows from linearity. Hence if the trace is* **finite***, we can extend it to the whole algebra* $\mathcal{M}$*.*

Example 1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} = \mathcal{B}(\mathcal{H})$. Given a maximal orthonormal system $\{\xi_\alpha\}$ in $\mathcal{H}$ we define*

$$\tau(a) = \sum_\alpha \langle a\xi_\alpha, \xi_\alpha \rangle,\ a \in \mathcal{B}(\mathcal{H})_+$$

The value of $\tau(a)$ does not depend on the particular choice of the maximal orthonormal system in $\mathcal{H}$ and $\tau : \mathcal{B}(\mathcal{H})_+ \to [0, \infty]$ is a semi-finite faithful normal trace on $\mathcal{B}(\mathcal{H})$. This is called the standard trace on $\mathcal{B}(\mathcal{H})$.

Definition 2.2.4. *A measure space (X, Σ, ν) is Maharam if it has the finite subset property, that is, for every $A \in \Sigma$ with $\nu(A) > 0$ there exists $B \in \Sigma$ such that $B \subset A$ and $0 < \nu(B) < \infty$.*

Remark 5. *We note that every sigma-finite measure space is Maharam.*

Example 2. *Let $\mathcal{H} = L_2(\nu)$, where (X, Σ, ν) is a Maharam measure space. On $L_2(\nu)$ we consider the von Neumann algebra $\mathcal{M} = L_\infty(\nu)$. If we define $\tau : L_\infty(\nu)^+ \to [0, \infty]$ by*

$$\tau(f) = \int_X f d\nu,\ 0 \leq f \in L_\infty(\nu),$$

then τ is a semi-finite normal trace on $L_\infty(\nu)$.

Example 3. *Consider a discrete group Γ and let $vN(\Gamma) \subset \mathcal{B}(\ell_2(\Gamma))$ be the associated von Neumann algebra generated by the left translations. Let τ_Γ be the canonical trace on $vN(\Gamma)$, defined as follows: $\tau_\Gamma(x) = \langle x(\delta_e), \delta_e \rangle$ for any $x \in vN(\Gamma)$, where $(\delta_g)_{g \in \Gamma}$ denotes the canonical basis of $l_2(\Gamma)$, and where e is the identity in Γ. This is a normal faithful normalized finite trace on $vN(\Gamma)$.*

2.3 Banach lattices and the lower p-estimate

We begin with some preliminary definitions to explain the concept of Banach lattices.

Definition 2.3.1. *A real vector space E which is ordered by some order relation $\leq$ is called a vector lattice if any two elements $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and the greatest lower bound denoted by $x \wedge y = \inf(x, y)$ and the following properties are satisfied.*

1. *$x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$*
2. *$0 \leq x$ implies $0 \leq tx$ for al $x \in E$ and $t \in \mathbb{R}_+$*

Definition 2.3.2. *A norm on a vector lattice E is called a lattice norm if*

$$|x| \leq |y| \text{ implies } ||x|| \leq ||y|| \text{ for } x, y \in E$$

where $|x| := x \vee (-x)$

Definition 2.3.3. *A Banach lattice is a real Banach space E endowed with and ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.*

Now we will introduce the concept of a complex Banach lattice. The complexification of a real Banach lattice E is the complex Banach space $E_{\mathbb{C}}$ whose elements are pairs $(x, y) \in E \times E$, with addition and scalar multiplication defined by $(x_0, y_0) + (x_1, y_1) := (x_0 + x_1, y_0 + y_1)$ and $(a + ib)(x, y) := (ax - by, ay + bx)$, and norm

$$||(x, y)|| := || \sup_{0 \leq \theta < 2\pi} (x \sin\theta + y \cos\theta)||$$

Definition 2.3.4. *A complex Banach lattice is an ordered complex Banach space $(E_{\mathbb{C}}, \leq)$ that arises as the complexification of a real Banach lattice E.*

Definition 2.3.5. *A Banach lattice E satisfies a lower p-estimate with constant K if, for any disjoint $x_1, ..., x_n$, we have that $||x_1 + ... + x_n||^p \geq K(\sum_i ||x_i||^p)$*

Note that here, $|x_1 + ... + x_n| = |x_1| \vee ... \vee |x_n|$, by Theorem 1.1.1(i) in [21]. The existence of a lower p-estimate and its connections to other properties of Banach lattices, have been studied extensively, see e.g. [20].

2.4 Banach Function Spaces

We proceed by introducing commutative and non-commutative function and sequence spaces. In general, our exposition follows [7] and the reader can consult that excellent survey paper for further information.

Let ν be the usual Lebesgue measure associated to the $Maharam$ measure space (X, Σ, ν). We also assume that (X, Σ, ν) is *localizable*, in other words, the measure algebra is a complete Boolean algebra. The space of all complex valued Σ-measurable functions on X is denoted by $L_0(\nu)$.

Definition 2.4.1. *A Banach function space on (X, Σ, ν) is an ideal $\mathcal{E} \subset L_0(\nu)$, that is, $\mathcal{E}$ is a linear subspace of $L_0(\nu)$ equipped with a norm, $||.||_{\mathcal{E}}$ such that $(\mathcal{E}, ||.||_{\mathcal{E}})$ is a Banach lattice with the additional property that if $f \in L_0(\nu), g \in \mathcal{E}$ and $|f| \leq| g|$ implies that $f \in \mathcal{E}$.*

For $f \in L_0(\nu)$ its distribution function $d_f : [0, \infty) \to [0, \infty]$ is defined by

$$d_f(\lambda) = \nu(\{x \in X : |f(x)| > \lambda\}),\ \lambda \geq 0$$

Note that d_f is decreasing and right-continuous. Now we define

$$S(\nu) = \{f \in L_0(\nu) : \exists \lambda_0 \geq 0 \text{ such that } d_f(\lambda_0) < \infty\}$$

If $f \in L_0(\nu)$, then $f \in S(\nu)$ if and only if f is bounded except on a set of finite measure. Therefore, $S(\nu)$ is an ideal in $L_0(\nu)$. For $f \in S(\nu)$ the *decreasing rearrangement* $\mu(f) : [0, \infty) \to [0, \infty]$ of $|f|$ is defined by:

$$\mu(f, t) = \inf\{\lambda \geq 0 : d_f(\lambda \leq t)\}, t \geq 0$$

Let $\mathcal{E}$ be a Banach function space on the Maharam measure space (X, Σ, ν).

Definition 2.4.2. *The Banach function space $\mathcal{E} \subset S(\nu)$ is called rearrangement invariant if $f \in \mathcal{E}, g \in S(\nu)$ and $\mu(g) = \mu(f)$ imply that $g \in \mathcal{E}$ and $||g||_{\mathcal{E}} = ||f||_{\mathcal{E}}$.*

Lemma 2.4.1. *If E is a Banach lattice, whose norm $\|\cdot\|$ satisfies a lower p-estimate with constant K, then E has an equivalent lattice norm $\|\cdot\|_0$, satisfying a lower p-estimate with constant 1, and such that $\|\cdot\| \leq \|\cdot\|_0 \leq K\|\cdot\|$. Moreover, if the norm $\|.\|$ is rearrangement invariant, then the same is true for $\|.\|_0$.*

Proof. For $x \in E$ define $\|x\|_0 = \sup\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$, where the supremum runs over all finite disjoint sequences $(x_i)_{i=1}^n$ s.t. $|x| = \vee_i |x_i|$. Clearly, $\|\cdot\|_0$ is a lattice quasi-norm, and $\|\cdot\| \leq \|\cdot\|_0 \leq K\|\cdot\|$. It remains to show that $\|\cdot\|$ is indeed a lattice norm. So consider $0 \leq y \leq x$. We show that $\|y\|_0 \leq \|x\|_0$. Suppose $x = \sum_i x_i = \vee_i x_i$ is a "finite disjoint decomposition." Then the elements $y_i = y \wedge x_i$ are disjoint, and, by the distributive law, Theorem 1.1.1(vii) in [21],

$$\textstyle\sum_i y_i = \vee_i (y \wedge x_i) = y \wedge x = y$$

Taking the infimum over all allowable finite collections (x_i), we get $\|y\|_0 \leq \|x\|_0$. Finally, assume that $(E, \|.\|)$ is rearrangement invariant (r.i.), then so is $(E, \|.\|_0)$ due to the fact that, if the sequences $(x_i)_{i=1}^n$ and $(x_i')_{i=1}^n$ are disjoint, and x_i and x_i' are equidistributed for any i, then $|x_1| \vee ... \vee |x_n|$ and $|x_1'| \vee ... \vee |x_n'|$ are equidistributed as well. $\square$

2.5 Symmetric Operator Spaces

Now let $\mathcal{H}$ denote a Hilbert Space and let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ denote a von Neumann algebra with an (n.s.f) trace τ. We denote this space by $(\mathcal{M}, \tau)$. Recall that a closed densely defined operator x on $\mathcal{H}$ is said to be *affiliated* with $\mathcal{M}$ if $xu = ux$ for any unitary $u \in \mathcal{M}'$, the commutant of $\mathcal{M}$. A closed densely defined operator x which is *affiliated* with $\mathcal{M}$ is called *τ-measurable* or just measurable if given $\epsilon > 0$, there exists an orthogonal projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subset \text{Dom}(x)$, $\tau(1-p) < \epsilon$ and $xp \in \mathcal{M}$. We denote $L^0(\mathcal{M}, \tau)$ as the set of all *τ-measurable* operators. The set $L^0(\mathcal{M}, \tau)$ is a $*$-algebra and given a self-adjoint operator $x \in L^0(\mathcal{M}, \tau)$, we denote by $e^x(.)$ its spectral measure. Now, recall that $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subset \mathcal{B}$ where $|x| = (x^*x)^{1/2}$ is the modulus of the operator x. Here, $\mathcal{B}$ denotes the Borel σ-algebra.

For any measurable operator x we define the *generalized singular numbers* by

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(e^{|x|}(\lambda, \infty)) \leq t\}, t > 0$$

Let $\mathcal{E}$ be a rearrangement invariant Banach function space on $[0, \infty)$, the symmetric operator space associated with $(\mathcal{M}, \tau)$ and $\mathcal{E}$ is defined by

$$\mathcal{E}(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \mu(x) \in \mathcal{E}\} \text{ and } ||x||_{\mathcal{E}(\mathcal{M},\tau)} = ||\mu(x)||_{\mathcal{E}}$$

It should be noted that $\mathcal{E}(\mathcal{M}, \tau)$ is a Banach space, which is Theorem 8.11 in [15].

Now consider the particular case of $\mathcal{E} = L^p$. This is a non-commutative L^p-space associated with $(\mathcal{M}, \tau)$. To be more specific, we define non-commutative L_p by letting $\mathcal{M}$ be a von Neumann algebra equipped with (n.s.f.) trace τ, and $L^0(\mathcal{M}, \tau)$ be the algebra of all τ-measureable operators affiliated with $\mathcal{M}$. Then we can define $L_p(\tau) = \{a \in L^0(\mathcal{M}, \tau) : \tau(|a|^p) < \infty\}$. It is a Banach space with respect to the norm $\|a\|_p = (\tau(|a|^p))^{1/p}$ for $1 \leq p < \infty$. We are primarily interested when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$ and τ is the standard trace on $\mathcal{B}(\mathcal{H})$. In this case, the construction of $L_p(\tau)$ yields the Schatten p-class $\mathcal{S}_p(\mathcal{H})$.

Also, one should note that, if $\mathcal{E}$ is a r.i. function space satisfying a lower p-estimate, then the same is true for the corresponding non-commutative Banach function space. The following lemma can easily be established:

Lemma 2.5.1. *Suppose $\mathcal{E}$ is a r.i. function or sequence space, satisfying the lower p-estimate with constant C. If $x_1, ..., x_n$ are elements of $\mathcal{E}(\mathcal{M}, \tau)$ whose left and right support projections are disjoint, then*

$$|| \sum_i x_i || \geq C(\sum_i ||x_i||^p)^{1/p}.$$

Chapter 3

Almost Commuting Hermitian Operators

3.1 Main Result

In this chapter we prove **Theorem 0.3.1** and **Corollary 0.3.2**. which are re-stated below:

Theorem 3.1.1. *Consider the non-commutative polynomial* $q(x,y) = xy - yx$*. Let* $(\mathcal{M}, \tau)$ *be a von Neumann algebra with a (n.s.f) trace* τ*. Let* $||.||$ *denote the usual operator norm and suppose* $(\mathcal{E}, \|\!|.|\!\|)$ *is a rearrangement invariant function space on* $[0, \tau(1))$*, satisfying the lower* p*-estimate for some* p *and constant* 1*, where* $1 \leq p < \infty$*. Assume that* $a \in \mathcal{M}_{sa}$ *is a self-adjoint operator such that* $||a|| \leq 1$*, and* $b \in \mathcal{E}(\mathcal{M}, \tau)$ *is an operator that satisfies* $\|\!|b|\!\| \leq 1$*. Then there exists operators* $\tilde{a}$*, with* $||\tilde{a}|| \leq ||a||$*, and an operator* $\tilde{b}$*, commuting with* $\tilde{a}$*, so that* $||a - \tilde{a}|| + \|\!|b - \tilde{b}|\!\| \leq K_1 \|\!|q(a,b)|\!\|^{\frac{1}{p+2}}$ *where the constant* $K_1 = K_1(p)$ *does not depend on the von Neumann Algebra. Moreover, if* b *is self-adjoint, then* $\tilde{b}$ *can also be chosen to be* *self-adjoint.*

As a result, we get the following corollaries,

Corollary 3.1.1. *Suppose that* $q(X,Y) = XY - YX$ *and let* $||.||$ *denote the usual operator norm. Let* $||.||_p$ *be the normalized Schatten p-norm, where* $1 \leq p < \infty$*. Assume that* $A \in M_N(\mathbb{C})$ *is a self-adjoint matrix such that* $||A|| \leq 1$*, and* $B \in M_N(\mathbb{C})$ *also satisfies* $||B||_p \leq 1$*. Then there exists commuting matrices* $\tilde{A}, \tilde{B} \in M_N(\mathbb{C})$*,* $q(\tilde{A}, \tilde{B}) = 0$*, such that* $||A - \tilde{A}||_p + ||B - \tilde{B}||_p \leq K_p ||AB + BA||_p^{1/(p+2)}$*. Here, the constant* K_p *does not depend on the dimension of our matrices. Moreover,* $\tilde{A}$ *is self-adjoint and if* B *is self-adjoint, then* $\tilde{B}$ *can also be chosen to be self-adjoint.*

Corollary 3.1.2. *Suppose that* $A \in M_n(\mathbb{C})$ *satisfies* $||A|| \leq 1$ *where* $||.||$ *is the usual operator norm. Let* $||.||_p$ *denote the normalized Schatten p-norm* $1 \leq p < \infty$*. Then there exists a normal matrix* N *such that* $||N - A||_p \leq K||AA^* - A^*A||_p^{1/(p+2)}$*, where* $K = K(p)$ *is a universal constant independent of the dimension of the matrix* A*.*

3.2 Preliminaries

Throughout this chapter we use the notation of **Theorem 3.1.1**. Now let

$$F(x) = \begin{cases} (1-x^2)^4 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Note that this function is three times continuously differentiable. Well known harmonic analysis results show that there exists a constant $C > 0$ so that $\hat{F}(t) \leq \min\{1, C/t^3\}$ for any real number t.

Now fix $n, m \in \mathbb{N}$ to be optimized later. Let $a \in \mathcal{M}_{sa}$ with $||a|| \leq 1$, and note that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and mutually orthogonal projections $Q_1, ..., Q_N$ summing to the identity, $\sum Q_j = \mathbf{1}$, and $a_1, ..., a_N \in \mathbb{R}$ such that $||a - \sum_{j=1}^{N} a_j Q_j|| < \epsilon$. Moreover, we can choose $\{a_j\}_{j=1}^{N} \subset \sigma(a) \subset [-1, 1]$ and we can assume $-1 \leq a_1 \leq ... \leq a_N \leq 1$. Hence, we may assume that $a = \sum_{j=1}^{N} a_j Q_j$.

Next, we define intervals:

$$I_1 = [-1, -1 + \tfrac{2}{mn}] \text{ and } I_k = (-1 + \tfrac{2(k-1)}{mn}, -1 + \tfrac{2k}{mn}], \text{ for } 2 \leq k \leq mn.$$

Let $P_i = \chi_{I_i}(a)$ be the corresponding spectral projection and let $\delta = 2/(mn)$. Also, for $k < 1$ or $k > mn$, set $I_k = \{\emptyset\}$, and $P_k = 0$.

3.3 Block Tri-diagonalization via Fourier Analysis

We start by perturbing the operator b to b' that is "block tri-diagonal" relative to a and does not differ much from b in norm. To this end, define:

Equation 2. $b' = \delta \int_{-\infty}^{\infty} e^{iat} b e^{-iat} \hat{F}(\delta t) dt$

Lemma 3.3.1. *The following hold for the operators defined above:*

1. $P_j b' P_i = 0$ *if* $|i - j| > 1$.

2. $\|b - b'\| \leq C_0 \delta^{-1} \|ab - ba\|$,*where* C_0 *is a universal constant.*

3. $\|b'\| \leq 2$ *if* $\delta \geq C_0 \|ab - ba\|$.

4. *If the operator* b *is self-adjoint, then so is* b'.

Proof. Note that Range P_k = span[Range $Q_r : a_r \in I_k$]. Therefore, (1) will be established if we show that, for $a_s \in I_i$ and $a_r \in I_j$, with $|i-j| > 1$, $\xi \in$ Range Q_s, and $\eta \in$Range Q_r we have $\langle b'\xi, \eta\rangle = 0$. By Fourier Inversion Formula,

$$\langle b'\xi, \eta\rangle = \delta \int \langle e^{iat} b e^{-iat}\xi, \eta\rangle \hat{F}(\delta t)dt$$

$$= \delta \int \langle b e^{-iat}\xi, e^{-iat}\eta\rangle \hat{F}(\delta t)dt$$

$$= \delta \int \langle b e^{-ia_s t}\xi, e^{-ia_r t}\eta\rangle \hat{F}(\delta t)dt$$

$$= \langle b\xi, \eta\rangle \int \delta e^{i(a_r - a_s)t} \hat{F}(\delta t)dt$$

$$= \langle b\xi, \eta\rangle F(\tfrac{a_r - a_s}{\delta})$$

Now if $|i-j| > 1$, then $|a_r - a_s| \geq \delta$ which implies that $F(\frac{a_r - a_s}{\delta}) = 0$. To establish (2), note that $\int \hat{F}(t)dt = F(0) = 1$, hence

$$b' - b = \delta \int_{-\infty}^{\infty} (e^{iat} b e^{-iat} - b) \hat{F}(\delta t)dt$$

which implies

$$\|b' - b\| \leq \delta \int_{-\infty}^{\infty} \|e^{iat} b e^{-iat} - b\| |\hat{F}(\delta t)|dt$$

Now let $g(t) = e^{iat} b e^{-iat} - b$. Since $g(0) = 0$, by the Mean Value Theorem, $\|g(t)\| \leq |t| \sup_{|s|\leq|t|} \|g'(s)\|$. Here, $g'(s) = ie^{ias}(ab - ba)e^{-ias}$, hence $\|g'(s)\| \leq \|ab - ba\|$. Thus, $\|b - b'\| \leq C_0 \delta^{-1} \|ab - ba\|$, where C_0 is the L_1 norm of the function $t \to t\hat{F}(t)$. (3) follows from (2) and the triangle inequality. Finally, (4) follows from the facts that the operators a and b are self-adjoint and that $\hat{F}$ is an even function. $\square$

3.4 Pinching

For $0 \leq i \leq m-1$, let

$$c_i{}^u = \sum_{j=0}^{n-1} P_{i+mj} b' P_{i+mj+1} \text{ and } c_i{}^l = \sum_{j=0}^{n-1} P_{i+mj+1} b' P_{i+mj},\ c_i = c_i{}^u + c_i{}^l$$

Lemma 3.4.1. *Suppose that $\|\!|.|\!\|$ is a norm associated to a symmetric operator space $\mathcal{E}(\mathcal{M}, \tau)$ and let $\{P_1, ..., P_n\}$ and $\{Q_1, ..., Q_n\}$ be two sets of mutually orthogonal projections in a von Neumann algebra $\mathcal{M}$ equipped with a (n.s.f) trace τ, then for any $x \in \mathcal{E}(\mathcal{M}, \tau)$ we have that:*

$$\|\!| \sum_{j=1}^{n} P_j x Q_j |\!\| \leq \|\!| x |\!\|.$$

Proof. Without loss of generality, we may assume that $x = \sum_{i,j} x_{i,j}$ where $x_{i,j} = P_i x Q_j$. Proceed by letting

$$E_n = \{(e_1, ..., e_n) : e_j \in \{-1, 1\}\}$$

and for $e \in E_n$ we let

$$U_e = \sum_{k=1}^{n} e_k P_k \text{ and } V_e = \sum_{k=1}^{n} e_k Q_k$$

Then we claim that the following equation holds:

Equation 3. $\sum_{j=1}^{n} P_j x Q_j = \frac{1}{2^n} \sum_{e \in E} U_e x V_e$.

To establish this identity, we observe that:

$$U_e x_{i,j} V_e = (\sum_{k=1}^{n} e_k P_k) x_{i,j} (\sum_{k=1}^{n} e_k Q_k)$$

$$= (\sum_{k=1}^{n} e_k P_k) P_i x Q_j (\sum_{k=1}^{n} e_k Q_k)$$

$$= e_i e_j P_i x Q_j = e_i e_j x_{i,j}$$

We must show that, for a given pair (i,j) we have that $\sum_{e\in E} e_i e_j = 2^n$ if $i=j$ and 0 if $i \neq j$. The former is clear. For the latter, observe that $|\{e \in E : e_i = e_j = 1\}| = |\{e \in E : e_i = e_j = -1\}| = |\{e \in E : e_i = 1, e_j = -1\}| = |\{e \in E : e_i = -1, e_j = 1\}| = 2^{n-2}$. □

Corollary 3.4.1. *We have that* $\|\sum_{i=0}^{m-1} c_i^u\| \leq \|b'\|$ *and* $\|\sum_{i=0}^{m-1} c_i^l\| \leq \|b'\|$.

Proof. Note that $\sum_{i=0}^{m-1} c_i^u = \sum_{s=0}^{mn-1} P_s b' P_{s+1}$ and $\sum_{i=0}^{m-1} c_i^l = \sum_{s=0}^{mn-1} P_{s+1} b' P_s$ and that the spectral projections $\{P_j\}$ have the property that $P_i P_j = 0$ for $i \neq j$. In other words, they are mutually orthogonal. Now just apply the previous lemma. □

Lemma 3.4.2. *There exists an index* i *so that* $\|c_i\| \leq 8m^{-1/p}$.

Proof. We have that $\|\sum_{i=0}^{m-1} {c_i}^l\| \leq \|b'\| \leq 2$. But $\|.\|$ satisfies the lower p-estimate with constant 1 so

$$\sum_{i=0}^{m-1} \|{c_i}^l\|^p \leq \|\sum_{i=0}^{m-1} {c_i}^l\|^p \leq \|b'\|^p \leq 2^p.$$

Hence $\|{c_i}^l\|^p > \frac{2^{p+1}}{m}$ for less than $m/2$ values of the indices i. In other words, $\|{c_i}^l\| \leq \frac{2^{1+1/p}}{m^{1/p}}$ for more than $m/2$ values of the indices i. Similarly, $\|{c_i}^u\| \leq \frac{2^{1+1/p}}{m^{1/p}}$ for more than $m/2$ values of i. Therefore, by the pigeonhole principle, there exists an index i such that the upper estimates for both $\|{c_i}^l\|$ and $\|{c_i}^u\|$ hold. By the triangle inequality, $\|c_i\| \leq \frac{8}{m^{1/p}}$. □

3.5 Proof of Main Result

We proceed by perturbing the operator b' to the operator $\tilde{b}$ which is "block diagonal" with larger blocks. To this end, for $0 \leq j \leq n$, define the intervals:

$$\tilde{I}_j = \bigcup_{s=i+m(j-1)+1}^{i+mj} I_s$$

and their corresponding spectral projections

$$\tilde{P}_j = \chi_{\tilde{I}_j}(a) = \sum_{s=i+m(j-1)+1}^{i+mj} P_s.$$

Denote the midpoint of $\tilde{I}_j$ by $\tilde{a}_j$, let $\tilde{b} = b' - c_i$, and "rough grain" the operator a to $\tilde{a} = \sum_{j=0}^{n} \tilde{a}_j \tilde{P}_j$. Note that since $\tilde{a}_j \in \mathbb{R}$ for all j and each of the operators $\tilde{P}_j$ are projections, we see that $\tilde{a}$ is also self-adjoint.

Now we prove a lemma that will be used to prove our main result.

Lemma 3.5.1. *For the operators we have defined, the following hold:*

(1) $\tilde{a}$ *commutes with* $\tilde{b}$.

(2) $||a - \tilde{a}|| \leq \frac{1}{n}$.

(3) $\|\tilde{b} - b\| \leq C_0 \delta^{-1} \|ab - ba\| + \frac{8}{m^{1/p}}$, *where* C_0 *is a universal constant.*

Proof. We begin proving (1) by first showing that for $j \neq k$, $\tilde{P}_j \tilde{b} \tilde{P}_k = 0$. We consider only the case when $j < k$, since $j > k$ is dealt in a similar fashion. Using the definition of $\tilde{P}_j$ and $\tilde{P}_k$ above and noting the fact that $P_s b' P_r = 0$ if $|s - r| > 1$, we get that $\tilde{P}_j b' \tilde{P}_k = 0$ if $j < k - 1$. Furthermore, in this situation $\tilde{P}_j c_i \tilde{P}_k = 0$. Thus, $\tilde{P}_j \tilde{b} \tilde{P}_k = \tilde{P}_j (b' - c_i) \tilde{P}_k = 0$. Now suppose that $j = k - 1$, then

$$\tilde{P}_j b' \tilde{P}_k = P_{i+mj} b' P_{i+mj+1} = P_{i+mj} c_i P_{i+mj+1},$$

hence

$$\tilde{P}_j \tilde{b} \tilde{P}_k = \tilde{P}_j (b' - c_i) \tilde{P}_k = 0,$$

which implies that

$$\tilde{b} = \sum_{j=0}^{n} \tilde{P}_j \tilde{b} \tilde{P}_j = \sum_{j=0}^{n} \tilde{P}_j b' \tilde{P}_j,$$

therefore, we have that $\|\tilde{b}\| \leq \|b'\|$ by Lemma 3.4.1 and it is clear that $\tilde{a} = \sum_{j=0}^{n} \tilde{a}_j \tilde{P}_j$ and $\tilde{b} = \sum_{j=0}^{n} \tilde{P}_j \tilde{b} \tilde{P}_j$ commute. For (2), note that if $a_s \in \tilde{I}_j$, then $|a_s - \tilde{a}_j| \leq n^{-1}$. Thus, $||a - \tilde{a}|| \leq n^{-1}$. To establish (3) we simply use the triangle inequality,

$$\|b - \tilde{b}\| \leq \|b - b'\| + \|c_i\| \leq C_0 \delta^{-1} \|ab - ba\| + \frac{8}{m^{1/p}}.$$

□

Now we use our previous lemma to give a concise proof of our main result.

Proof of Theorem 3.1.1.

Proof. It remains to show that $||a - \tilde{a}|| + \|b - \tilde{b}\| \leq K_p \|ab - ba\|^{\frac{1}{p+2}}$. Let $\epsilon = \|ab - ba\|^{\frac{1}{p+2}}$ and let $n = \lceil 1/\epsilon \rceil$ and $m = \lceil (\frac{pn}{16} \|ab - ba\|)^{-p/(p+1)} \rceil$. Then clearly $||a - \tilde{a}|| \leq \epsilon$. Now by (3) of Lemma 3.5.2 above we have that

$$\|b - \tilde{b}\| \leq C_0 \delta^{-1} \|ab - ba\| + 8m^{-1/p}$$

$$= C_0 \frac{mn}{2} \|ab - ba\| + 8m^{-1/p}.$$

and by our choice of n and m we have that $||a - \tilde{a}|| + \|b - \tilde{b}\| \leq K(p) \|ab - ba\|^{\frac{1}{p+2}}$ where $K(p)$ is a constant that does not depend on our von Neumann algebra. Now if b is self-adjoint, $b = b^* \Rightarrow b' = b'^* \Rightarrow ({c_i}^u)^* = {c_i}^l$ and $({c_i}^l)^* = {c_i}^u \Rightarrow c_i = {c_i}^* \Rightarrow \tilde{b} = \tilde{b}^*$. Hence, if we have that a, b are self-adjoint, then their commuting approximates are also self-adjoint.

□

Proof of Corollary 3.1.2.: Let us write $A = B + iC$ where B, C are self-adjoint contractions. Note that $[A, A^*] = 2i[B, C]$ Now, by **Theorem 3.1.1**. there exist commuting self-adjoint matrices $\tilde{B}, \tilde{C}$ such that $||B - \tilde{B}||_p + ||C - \tilde{C}||_p \leq K||[B, C]||_p^{1/(p+2)} \leq K_2||[A, A^*]||_p^{1/(p+2)}$ so if we set $N = \tilde{B} + i\tilde{C}$ then we have that $||N - A||_p \leq K_2||AA^* - A^*A||_p^{1/(p+2)}$, where $K_2 = K_2(p)$ is a universal constant independent of the dimension of the matrix A. □

Chapter 4

Almost Anti-Commuting Hermitian Operators

4.1 Main Result

In this chapter we prove the following theorem:

Theorem 4.1.1. *Consider the non-commutative polynomial* $q(x, y) = xy + yx$ *and let* $(\mathcal{M}, \tau)$ *be a von Neumann algebra with a (n.s.f) trace* τ*. Let* $||.||$ *denote the usual operator norm and suppose* $(\mathcal{E}, |||.|||)$ *is a rearrangement invariant Banach function space on* $[0, \tau(1))$*, satisfying the lower* p*-estimate for some* p *and constant* 1*, where* $1 \leq p < \infty$*. Assume that* $a \in \mathcal{M}_{sa}$ *is a self-adjoint operator such that* $||a|| \leq 1$*, and* $b \in \mathcal{E}(\mathcal{M}, \tau)$ *is an operator with* $|||b||| \leq 1$*. Then there exists operators* $\tilde{a}$*, with* $||\tilde{a}|| \leq ||a||$*, and an operator* $\tilde{b}$*, anti-commuting with* $\tilde{a}$*, that is,* $q(\tilde{a}, \tilde{b}) = 0$*, with the property that* $||a - \tilde{a}|| + |||b - \tilde{b}||| \leq K_1 |||q(a, b)|||^{\frac{1}{p+2}}$*. Here, the constant* $K_1 = K_1(p)$ *does not depend on our von Neumann Algebra. Moreover, if* b *is* *self-adjoint, then $\tilde{b}$ can also be chosen to be self-adjoint.*

As a corollary, we get

Corollary 4.1.1. *Suppose that $q(X,Y) = XY + YX$ and let $||.||$ denote the usual operator norm. Let $||.||_p$ be the normalized Schatten p-norm, where $1 \leq p < \infty$. Assume that $A \in M_N(\mathbb{C})$ is a self-adjoint matrix such that $||A|| \leq 1$, and $B \in M_N(\mathbb{C})$ also satisfies $||B||_p \leq 1$. Then there exists anti-commuting matrices $\tilde{A}, \tilde{B} \in M_N(\mathbb{C})$, $q(\tilde{A}, \tilde{B}) = 0$, such that $||A - \tilde{A}||_p + ||B - \tilde{B}||_p \leq K_p ||AB + BA||_p^{1/(p+2)}$. Here, the constant K_p does not depend on the dimension of our matrices. Moreover, $\tilde{A}$ is self-adjoint and if B is self-adjoint, then $\tilde{B}$ can also be chosen to be self-adjoint.*

4.2 Preliminaries

Throughout this chapter we use the notation in **Theorem 4.1.1**. Now let

$$F(x) = \begin{cases} (1-x^2)^4 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Note that this function is three times continuously differentiable. Well known harmonic analysis results show that there exists a constant $C > 0$ so that $\hat{F}(t) \leq \min\{1, C/t^3\}$ for any real number t.

Now fix $n, m \in \mathbb{N}$ to be optimized later. Let $a \in \mathcal{M}_{sa}$ with $||a|| \leq 1$, and note that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and mutually orthogonal projections $Q_1, ..., Q_N$ summing to the identity, $\sum Q_j = \mathbf{1}$, and $a_1, ..., a_N \in \mathbb{R}$ such that $||a - \sum_{j=1}^N a_j Q_j|| < \epsilon$. Moreover, we can choose $\{a_j\}_{j=1}^N \subset \sigma(a) \subset [-1, 1]$ and we can assume $-1 \leq a_1 \leq ... \leq a_N \leq 1$. Hence, we may assume that $a = \sum_{j=1}^N a_j Q_j$.

Define the intervals,

$$I_0 = (-\tfrac{1}{2mn}, \tfrac{1}{2mn})$$

$$I_i = (-\tfrac{1}{2mn} + \tfrac{i}{mn}, \tfrac{1}{2mn} + \tfrac{i}{mn}] \text{ for } i < 0$$

$$I_i = [-\tfrac{1}{2mn} + \tfrac{i}{mn}, \tfrac{1}{2mn} + \tfrac{i}{mn}) \text{ for } i > 0$$

where $-mn \leq i \leq mn$. Now let $P_i = \chi_{I_i}(a)$ be the corresponding spectral projection of the operator a and let $\delta = 2/mn$.

4.3 Skew Block Tri-diagonalization via Fourier Transform

We begin by perturbing the operator b to a skew block-tridiagonal operator b' that does not "differ" much from b. To accomplish this, we define the operator:

Equation 4. $b' = \delta \int_{-\infty}^{\infty} e^{iat} b e^{iat} \hat{F}(\delta t) dt$

Lemma 4.3.1. *The following hold properties hold for the operator b':*

1. $P_j b' P_i = 0$ *if* $|i + j| > 1$.

2. $\|b - b'\| \leq C_0 \delta^{-1} \|ab + ba\|$,*where* C_0 *is a universal constant.*

3. $\|b'\| \leq 2$ *if* $\delta \geq C_0 \|ab + ba\|$.

4. *If the operator* b *is self-adjoint, then so is* b'.

Proof. Note that Range P_k = span[Range $Q_r : a_r \in I_k$]. Therefore, (1) will be established if we show that, for $a_s \in I_i$ and $a_r \in I_j$, with $|i+j| > 1$, $\xi \in$ Range Q_s, and $\eta \in$Range Q_r we have $\langle b'\xi, \eta\rangle = 0$. By Fourier Inversion Formula,

$$\langle b'\xi, \eta\rangle = \delta \int \langle e^{iat} b e^{iat}\xi, \eta\rangle \hat{F}(\delta t)dt$$

$$= \delta \int \langle b e^{iat}\xi, e^{-iat}\eta\rangle \hat{F}(\delta t)dt$$

$$= \delta \int \langle b e^{ia_s t}\xi, e^{-ia_r t}\eta\rangle \hat{F}(\delta t)dt$$

$$= \langle b\xi, \eta\rangle \int \delta e^{i(a_r+a_s)t} \hat{F}(\delta t)dt$$

$$= \langle b\xi, \eta\rangle F(\tfrac{a_r+a_s}{\delta})$$

Now if $|i+j| > 1$, then $|a_r + a_s| \geq \delta$ which implies that $F(\frac{a_r+a_s}{\delta}) = 0$. To establish (2), note that $\int \hat{F}(t)dt = F(0) = 1$, hence

$$b' - b = \delta \int_{-\infty}^{\infty} (e^{iat} b e^{iat} - b)\hat{F}(\delta t)dt$$

which implies

$$\|b' - b\| \leq \delta \int_{-\infty}^{\infty} \|e^{iat} b e^{iat} - b\| |\hat{F}(\delta t)| dt$$

Now let $g(t) = e^{iat} b e^{iat} - b$. Since $g(0) = 0$, by the Mean Value Theorem, $\|g(t)\| \leq |t| \sup_{|s|\leq|t|} \|g'(s)\|$. Here, $g'(s) = ie^{ias}(ab+ba)e^{ias}$, hence $\|g'(s)\| \leq \|ab+ba\|$. Thus, $\|b - b'\| \leq C_0\delta^{-1}\|ab+ba\|$, where C_0 is the L_1 norm of the function $t \to t\hat{F}(t)$. (3) follows from (2) and the triangle inequality. Now to establish (4) let us recall that the function $F(x)$ is an even function, hence so is $\hat{F}(\xi)$. So we have that

$$(b')^* = (\delta \int_{-\infty}^{\infty} e^{iat} b e^{iat} \hat{F}(\delta t)dt)^* = \delta \int_{-\infty}^{\infty} (e^{iat} b e^{iat} \hat{F}(\delta t))^* dt$$

$$= \delta\int_{-\infty}^{\infty}(e^{iat}\hat{F}(\delta t))^*(e^{iat}b)^*dt = \delta\int_{-\infty}^{\infty}\overline{\hat{F}(\delta t)}e^{-iat}be^{-iat}dt$$

Now just let $t \to -t$ and since F is an even function we get that $(b')^* = b'$ as claimed.

□

4.4 Pinching

Now we make b' into a skew block-diagonal operator $\tilde{b}$ which also does not "differ" much from the operator b. To this end, note that

$$b' = \sum_k P_k b' P_{-k} \text{ (main skew diagonal)}$$

$$+\sum_k P_k b' P_{1-k} \text{ (upper skew diagonal)}$$

$$+\sum_k P_k b' P_{-1-k} \text{ (lower skew diagonal)}$$

where $-mn \leq k \leq mn$. For convienence, we set $P_i = 0$ for $i \notin \{-mn, ..., mn\}$. Now we use "symmetric pinching" to get rid of some of the blocks above and below the main skew-diagonal. So for $1 \leq i \leq m$ let

$$c_i^u = \sum_{j=0}^{n}(P_{i+jm}b'P_{1-(i+jm)} + P_{1-(i+jm)}b'P_{i+jm}) \text{ and}$$
$$c_i^l = \sum_{j=0}^{n}(P_{i+jm-1}b'P_{-(i+jm)} + P_{-(i+jm)}b'P_{i+jm-1})$$

and for $1 \leq i \leq m$, set $c_i = c_i^u + c_i^l$. Note that $\sum_i c_i^l = \sum_k P_k b' P_{-1-k}$ (the lower skew diagonal) and $\sum_i c_i^u = \sum_k P_k b' P_{1-k}$ (the upper skew-diagonal). Henceforth, we assume that $\delta \geq C_0 |||ab + ba|||$.

Lemma 4.4.1. *There exists an index i such that $\|c_i\| \leq \frac{8}{m^{1/p}}$.*

Proof. This follows from the same arguement used to establish **Lemma 3.4.1** and **Corollary 3.4.1**. We have that $\| \sum_{i=1}^{m} {c_i}^l \| \leq \|b'\| \leq 2$. But $\|.\|$ satisfies the lower p-estimate with constant 1 so

$$\sum_{i=1}^{m} \|{c_i}^l\|^p \leq \| \sum_{i=0}^{m-1} {c_i}^l \|^p \leq \|b'\|^p \leq 2^p$$

Hence $\|{c_i}^l\|^p > \frac{2^{p+1}}{m}$ for less than $m/2$ values of i. In other words, $\|{c_i}^l\| \leq \frac{2^{1+1/p}}{m^{1/p}}$ for more than $m/2$ values of i. Similarly, $\|{c_i}^u\| \leq \frac{2^{1+1/p}}{m^{1/p}}$ for more than $m/2$ values of i. Therefore, by the pigeonhole principle, there exists an index i such that the upper estimates for $\|{c_i}^l\|$ and $\|{c_i}^u\|$ hold. By the triangle inequality, $\|c_i\| \leq \frac{8}{m^{1/p}}$. □

4.5 Proof of Main Result

We keep the same notation that we have been using. Let $F_0 = \{1-i, ..., i-1\}$ and for $j > 0$ we let $F_j = \{i + m(j-1), ..., i + mj - 1\}$, and $F_{-j} = -F_j$. Now for $-n \leq j \leq n$ we define the intervals,

$$\tilde{I}_j = \bigcup_{s \in F_j} I_s$$

Consider the corresponding spectral projections $\tilde{P}_j = \chi_{\tilde{I}_j}(a) = \sum_{s \in F_j} P_s$. Denote the mid-point of $\tilde{I}_j$ by $\tilde{a}_j$ and note that $-\tilde{I}_{-j} = \tilde{I}_j$ so we have that $-\tilde{a}_j = \tilde{a}_{-j}$. Let $\tilde{b} = b' - c_i$. We "rough grain" the operator a to $\tilde{a}$ where $\tilde{a} = \sum_{j=-n}^{n} \tilde{a}_j \tilde{P}_j$.

Lemma 4.5.1. *We claim that:*

1. $\|a - \tilde{a}\| \leq n^{-1}$
2. $\|b - \tilde{b}\| \leq C_0 \delta^{-1} \; \|ab + ba\| + 8m^{-1/p}$
3. *$\tilde{a}$ anti-commutes with $\tilde{b}$.*
4. *If b is self-adjoint, then so is $\tilde{b}$.*

Proof. To prove (1), notice that if $a_s \in \tilde{I}_j$, then $|a_s - \tilde{a}_j| \leq n^{-1}$. Thus, we have that $||a - \tilde{a}|| \leq n^{-1}$.

For (2), we see that

$$\|b - \tilde{b}\| \leq \|b - b'\| + \|c_i\| \leq C_0 \delta^{-1} \|ab + ba\| + 8m^{-1/p}.$$

We begin proving (3) by showing first that, for $k \neq -j$, $\tilde{P}_j \tilde{b} \tilde{P}_k = 0$. Recall that $\tilde{P}_j = \sum_{r \in F_j} P_r$ and that $\tilde{P}_j b' \tilde{P}_k = \sum_{r \in F_j, l \in F_k} P_r b' P_l$. Now, $P_r b' P_l = 0$ if $|r + l| > 1$ so that we clearly have that $\tilde{P}_j b' \tilde{P}_k = 0$ for $|j + k| > 1$. Furthermore, in this situation $\tilde{P}_j c_i \tilde{P}_k = 0$. So now we need only to consider the case where $k = 1 - j$ and the case where $k = -1 - j$ is done similarly. Then,

$$\tilde{P}_j b' \tilde{P}_{1-j} = \sum_{r \in F_j, l \in F_{1-j}} P_r b' P_l = P_{r_0} b' P_{l_0} = \tilde{P}_j c_i \tilde{P}_{1-j}$$

where $r_0 = i + m(j-1)$, $l_0 = -(i + m(j-1) - 1)$. Hence, we have shown that $\tilde{P}_j \tilde{b} \tilde{P}_k = \tilde{P}_j (b' - c_i) \tilde{P}_k = 0$ for $j \neq -k$ and this leads us to conclude that $\tilde{b} = \sum_k \tilde{P}_k \tilde{b} \tilde{P}_{-k}$, therefore

$$\tilde{a}\tilde{b} = (\sum_s \tilde{a}_s \tilde{P}_s)(\sum_r \tilde{P}_r \tilde{b} \tilde{P}_{-r}) = \sum_s \tilde{a}_s \tilde{P}_s \tilde{b} \tilde{P}_{-s}$$

$$= (\sum_s \tilde{P}_s \tilde{b} \tilde{P}_{-s})(- \sum_s \tilde{a}_{-s} \tilde{P}_{-s})$$

$$= -\tilde{b}\tilde{a}$$

which is the desired anti-commutation. To establish (4) we note that by Lemma 4.3.1. we know that if b is self-adjoint, then the same holds for b'. Now, $\tilde{b} = b' - c_i$. But $(c_i^u)^* = (c_i^l)$ and $(c_i^l)^* = (c_i^u)$ so $c_i^* = c_i$ and we deduce that $\tilde{b}$ is also self-adjoint if the operator b is. □

Proof of Theorem 4.1.1. It remains to show that $||a - \tilde{a}||$, $\|b - \tilde{b}\| \leq K_2 \|ab + ba\|^{\frac{1}{p+2}}$ where $K_2 = K_2(p)$ does not depend on the von Neumann algebra. Let $\epsilon = \|ab + ba\|^{\frac{1}{p+2}}$ and let $n = \lceil 1/\epsilon \rceil$. Then clearly $||a - \tilde{a}|| \leq \epsilon$. Now by (2) of **Lemma 4.5.1**. above we have that

$$\|b - \tilde{b}\| \leq C_0 \delta^{-1} \|ab + ba\| + 8m^{-1/p}$$

$$= C_0 \frac{mn}{2} \|ab + ba\| + 8m^{-1/p}$$

Then for $m = \lceil (\frac{pn}{16} \|ab + ba\|)^{-p/(p+1)} \rceil$, we have that $||a - \tilde{a}|| + \|b - \tilde{b}\| \leq K_2 \|ab + ba\|^{\frac{1}{p+2}}$ where $K_2 = K_2(p)$ is a constant that does not depend on our von Neumann algebra.

□

Chapter 5

Operators that Almost Commute up to a Factor

5.1 Main Result

In this chapter we consider the approximate version of the operator equation, $ab = \omega ba$, where $\omega \in \mathbb{C}$ and a is a unitary operator. This corresponds to the polynomial $q(x, y) = xy - \omega yx$ in our equivalent formulation of the problem which was defined in the introduction. In this setting, we have established:

Theorem 5.1.1. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with an (n.s.f) trace τ. Let $||.||$ denote the usual operator norm and suppose $(\mathcal{E}, |||.|||)$ is a rearrangement invariant Banach function space on $[0, \tau(1))$, satisfying the lower p-estimate for some p and constant 1, where $1 \leq p < \infty$. Assume that $a \in \mathcal{M}$ is a unitary operator, and $b \in \mathcal{E}(\mathcal{M}, \tau)$ satisfies $|||b||| \leq 1$. Let ω be a root of unity and consider the non-commutative polynomial $q(x, y) = xy - \omega yx$.*

Then there exist operators $\tilde{a}, \tilde{b}$ *with* $q(\tilde{a}, \tilde{b}) = 0$ *and* $||a - \tilde{a}|| + \|b - \tilde{b}\| \leq K_{p,\omega} \|q(a,b)\|^{1/(p+2)}$ *where* $K_{p,\omega}$ *is a constant that is independent of the von Neumann algebra. Moreover,* $\tilde{a}$ *is a unitary operator.*

Remark 6. *1. In the general case, if* $b \neq 0$ *and* a *is a unitary, and* $ab = \omega ba$*, then* $|\omega| = 1$*. Indeed, then* $b = \omega a^* ba$ *and taking norms of both sides yields the result. To show that* ω *no longer needs to be a root of unity, let* $(e_i)_{i \in \mathbb{Z}}$ *be the canonical basis in* $\ell_2(\mathbb{Z})$*. Consider the unitaries* a *and* b*, defined via* $ae_i = e_{i-1}$ *and* $be_i = \omega^i e_i$*.*

2. In a similar fashion, one can show that, if a *and* b *are* $n \times n$ *matrices with* a *unitary and* b *not nilpotent, then* ω *is a root of unity. This can seen by considering the fact that If* $ab = \omega ba$*, then* $b = \omega a^* ba$ *then clearly the spectra of* b *and* $\omega a^* ba$ *coincide, hence* $\sigma(b) = \omega\sigma(b) = \omega^2\sigma(b) = ...$ *But the spectrum of a matrix is finite, hence the orbit of any non-zero member of* $\sigma(b)$ *under multiplication by* ω *is finite. This happens if and only if* ω *is a root of unity. If* b *is allowed to be nilpotent, then* ω *does not need to be a root of unity. Indeed one can consider* 2×2 *matrices* a *and* b *of the form:*

$$a = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$$

and

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3. It should be noted that, if A *and* B *are two matrices,* A *is normal,* B *non-singular, and* $AB = \lambda BA \neq 0$*, then* λ *is a root of unity. Indeed, then, by Theorem 2.5 of [5]* AB *is not nilpotent. Now apply Theorem 2.4 of the same paper.*

5.2 Preliminaries

Throughout this chapter we use the notation of **Theorem 5.1.1**. So let $\omega = e^{2i\pi\frac{L_0}{M_0}}$ where $L_0, M_0 \in \mathbb{N}$ are such that $(L_0, M_0) = 1$ and $0 \leq L_0 \leq M_0 - 1$. Now, as in the self adjoint case, for any unitary $a \in \mathcal{M}$ and for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$, $a_1, ..., a_N \in \mathbb{T}$ and mutually orthogonal projections $\{Q_j\}_{j=1}^N$ that sum to the identity, such that we have $||a - \sum_{j=1}^N a_j Q_j|| < \epsilon$. Moreover, we may choose $a_j \in \sigma(a) \subset \mathbb{T}$. Hence, we may assume that $a = \sum_{s=1}^N a_s Q_s$. Let M, m, and n be a fixed positive integers such that $M = M_0 mn$ and let $\delta = 2\pi/M$. Now represent the unit circle as a disjoint union of arcs of length δ, that is

$$I_k = \{e^{it} : t \in [(k-1)\delta, k\delta)\} \text{ for } 1 \leq k \leq M$$

and let $P_k = \chi_{I_k}(a)$ be the corresponding spectral projection of a. As in Chapters 3 and 4 the main step consists of perturbing b slightly, to obtain a "shifted" block tri-diagonal operator b'. To this end, define the function $F_\delta(x)$ on $[-\pi, \pi]$ identified with the unit circle $\mathbb{T}$ by setting:

$$F_\delta(x) = \begin{cases} (1-(x/\delta)^2)^3 & |x| \leq \delta \\ 0 & |x| > \delta \end{cases}$$

Now for any $k \in \mathbb{Z}$,

$$\hat{F}_\delta(k) = \int_{-\pi}^{\pi} (1-(x/\delta)^2)^3 e^{-ikx} dx = \delta \int_{\mathbb{R}} (1-y^2)^3 e^{-ik\delta y} dy$$

$$= \delta \hat{F}(\delta k) \leq \begin{cases} \delta & |k| \leq \delta^{-1} \\ C_0 \delta^{-2} k^{-3} & |k| \geq \delta^{-1} \end{cases}$$

Here, C_0 is a constant independent of k and δ.

5.3 Block Tri-Diagonalization via Fourier Series

We proceed, as in the self-adjoint case, by perturbing the operator b to an operator b' that consists of two block tri-diagonal pieces, and has the property that b' not "differ" much from the operator b. We use a similar method for achieving our goal with the modification that we use Fourier Series rather than the Fourier Transform. So we define:

Equation 5. $b' = \sum_{k \in \mathbb{Z}} \hat{F}_\delta(k) \omega^{-k} a^{-k} b a^k$

Lemma 5.3.1. *The following properties hold for the operators just defined:*

1. $P_j b' P_i = 0$ *if* $(i-j) \mod M \notin \{L_0mn - 1, L_0mn, L_0mn + 1\}$.

2. $\|b - b'\| \leq C_1 \delta^{-1} \|ab - \omega ba\|$, *where* C_1 *is a universal constant.*

3. $\|b'\| \leq 2$ *if* $\delta \geq C_0 \|ab - \omega ba\|$.

Proof. Note that RangeP_k = span [Range Q_r : $a_r \in I_k$]. Therefore, (1) will be established if we show that, for $a_s = e^{iu_s} \in I_i$ and $a_r = e^{iu_r} \in I_j$, with the property that $i - j \mod M \notin \{L_0mn - 1, L_0mn, L_0mn + 1\}$ we have that $\langle b'\xi, \eta \rangle = 0$ for $\xi \in$ Range Q_s and $\eta \in$ Range Q_r.

Now, by Fourier inversion,

$$\langle b'\xi, \eta \rangle = \sum_{k \in \mathbb{Z}} \hat{F}_\delta(k) \langle \omega^{-k} a^{-k} b a^k \xi, \eta \rangle$$

$$= \sum_{k \in \mathbb{Z}} \hat{F}_\delta(k) \langle \omega^{-k} b a^k \xi, a^k \eta \rangle$$

$$= \sum_{k \in \mathbb{Z}} \hat{F}_\delta(k) \langle b \omega^{-k} (a_s)^k \xi, (a_r)^k \eta \rangle$$

$$= \langle b\xi, \eta \rangle \sum_{k \in \mathbb{Z}} \hat{F}_\delta(k) (a_s / \omega a_r)^k$$

$$= \langle b\xi, \eta \rangle F_\delta(u_{sr} - 2\pi \tfrac{L_0}{M_0})$$

where $u_{sr} = u_s - u_r$. So we have that $\langle b'\xi, \eta \rangle = 0$ if $|u_s - u_r - 2\pi \frac{L_0}{M_0}| \geq \delta$ where the addition is modulo 2π.

On the other hand, when $|u_s - u_r - 2\pi \frac{L_0}{M_0}| < \delta$, we have two possible cases:

1. $-\frac{2\pi}{M_0 mn} < u_{sr} - 2\pi \frac{L_0}{M_0} < \frac{2\pi}{M_0 mn}$
2. $-\frac{2\pi}{M_0 mn} < u_{sr} - 2\pi \frac{L_0}{M_0} + 2\pi < \frac{2\pi}{M_0 mn}$

In the first situation we get

$$\delta(L_0 mn - 1) < u_s - u_r < \delta(L_0 mn + 1)$$

which implies that $i - j \in \{L_0 mn - 1, L_0 mn, L_0 mn + 1\}$. In the second scenario,

$$\delta(L_0 mn - 1) < u_s - u_r - 2\pi \tfrac{L_0}{M_0} < \delta(L_0 mn + 1)$$

which implies that $i - j + M \in \{L_0 mn - 1, L_0 mn, L_0 mn + 1\}$. Note that $1 \leq i, j \leq M$, hence $-(M-1) \leq i - j \leq M - 1$. To establish (2), note that $\sum_{k\in\mathbb{Z}} \hat{F}_\delta(k) = F_\delta(0) = 1$. Therefore,

$$\|b' - b\| = \| \textstyle\sum_{k\in\mathbb{Z}} \hat{F}_\delta(k)(\omega^{-k} a^{-k} b a^k - b)\|$$

$$\leq \textstyle\sum_{k\in\mathbb{Z}} |\hat{F}_\delta(k)| \|\omega^{-k} a^{-k} b a^k - b\|$$

Next we show that $\|\omega^{-k} a^{-k} b a^k - b\| \leq |k| \|ab - \omega ba\|$. It is enough to consider $k > 0$. Write $\omega^{-k} a^{-k} b a^k - b = \sum_{j=1}^{k} (\omega^{-j} a^{-j} b a^j - \omega^{-(j-1)} a^{-(j-1)} b a^{j-1})$. Since a is unitary, we have that

$$\|(\omega^{-j} a^{-j} b a^j - \omega^{-(j-1)} a^{-(j-1)} b a^{j-1})\|$$

$$= \|\omega^{-j}(a^{-j} b a^j - \omega a^{-(j-1)} b a^{j-1})\|$$

$$= \|a^{-j} b a^j - \omega a^{-(j-1)} b a^{j-1}\| = \|a^{-j}(ba - \omega ab) a^{j-1}\| = \|ab - \omega ba\|$$

The triangle inequality implies that

$$\|\omega^{-k}a^{-k}ba^k - b\| \leq |k|\|ab - \omega ba\|$$

Therefore,

$$\|b' - b\| \leq \|ab - \omega ba\| \sum_{k\in\mathbb{Z}} |\delta\hat{F}(\delta k)||k|$$

Hence, we have the following estimate,

$$\|b' - b\| \leq \|ab - \omega ba\| \times (\delta \sum_{|k|\leq\delta^{-1}} |k| + \delta^{-2} \sum_{|k|>\delta^{-1}} |k||k|^{-3})$$

$$\leq C_1\delta^{-1}\|ab - \omega ba\|$$

where C_1 is a universal constant. This proves (2). Now (3) follows from (2) and the triangle inequality. □

5.4 Pinching

Now we use a surgery procedure to take the tri-diagonal pieces of b' and make them block diagonal. As in the self-adjoint case, we accomplish this by a "pinching." To this end, note that by the previous lemma, by identifying $0 = 1 - 1$ with M and adding integers modulo M, we can write

$$b' = \sum_{s=1}^{M} P_{s+L_0mn}b'P_s + \sum_{s=1}^{M}(P_{s+L_0mn-1}b'P_s + P_{s+L_0mn}b'P_{s-1})$$

For $1 \leq i \leq n$, set

$$c_i^u = \sum_{j=0}^{M_0m-1} P_{L_0mn+jn+i-1}b'P_{jn+i},\ c_i^\ell = \sum_{j=0}^{M_0m-1} P_{L_0mn+jn+i}b'P_{jn+i-1},\ c_i = c_i^u + c_i^\ell.$$

Then

$$\sum_{i=1}^{n} c_i^u = \sum_{s=1}^{M} P_{L_0mn+s-1} b' P_s \text{ and } \sum_{i=1}^{n} c_i^\ell = \sum_{s=1}^{M} P_{L_0mn+s} b' P_{s-1}.$$

We combine **Lemma 3.4.1** with the lower p-estimate, to get

$$2^p \geq \|b'\|^p \geq \|\sum_{i=1}^{n} c_i^u\|^p \geq \sum_{i=1}^{n} \|c_i^u\|^p,$$

and similarly, $2^p \geq \|b'\|^p \geq \sum_{i=1}^{n} \|c_i^\ell\|^p$. Thus, as in the previous proof of almost commuting self-adjoint operators, we use the pigeonhole principle to deduce that there exists and index i so that $\|c_i\| \leq 8n^{-1/p}$.

5.5 Proof of Theorem 5.1.1.

We proceed by defining the intervals F_j $(0 \leq j \leq M_0m - 1)$ as $F_j = [jn + i, (j+1)n + i - 1]$. Here we are adding integers modulo M, so $F_{M_0m-1} = \{M_0mn - n + i, \ldots, M, 1, \ldots, i-1\}$. Let $\tilde{P}_j = \sum_{s \in F_j} P_s$ be the spectral projection of a, corresponding to the arc $\tilde{I}_j = \cup_{s \in F_j} I_s$. That is,

$$\tilde{I}_j = \left\{ e^{\iota t} : t \in \left[(jn + i - 1)\delta, ((j+1)n + i - 1)\delta \right) \right\}$$

These arcs are mutually disjoint, and their union is the whole unit circle $\mathbb{T}$. The length of each of these arcs equals $\delta n = 2\pi M_0^{-1} m^{-1}$.

Finally, let us define

$$\tilde{b} = \sum_{j=0}^{M_0m-1} \tilde{P}_{j+L_0m} b' \tilde{P}_j$$

Lemma 5.5.1. *The following equation holds,* $b' = \tilde{b} + c_i$.

Once the **Lemma 5.5.1**. is established, we are almost done. Set

$$\tilde{a} = \sum_{j=0}^{M_0 m-1} \tilde{\alpha}_j \tilde{P}_j,$$

where $\tilde{\alpha}_j$ is the midpoint of the arc $\tilde{I}_j$. More specifically,

$$\tilde{\alpha}_j = \exp(\iota\delta((j+1/2)n + i - 1)),$$

hence in particular,

$$\frac{\tilde{\alpha}_{j+L_0 m}}{\tilde{\alpha}_j} = \exp\big(\iota\delta L_0 mn\big) = \exp(2\pi i \frac{L_0}{M_0}) = \omega.$$

In other words, we can use the functional calculus for unitary operators and set $\tilde{a} = f(a)$, where $f = \sum_j \tilde{\alpha}_j \chi_{\tilde{I}_j}$. Moreover, $a = g(a)$, where g is the identity function, hence,

$$\|a - \tilde{a}\| \leq \|f - g\|_\infty = \frac{\pi}{M_0 m}.$$

By the triangle inequality,

$$\|b - \tilde{b}\| \leq \|b - b'\| + \|c_i\| \leq \frac{C_0 M_0 mn}{2\pi}\|ab - \omega ba\| + 8n^{-1/p}.$$

Putting everything together, we have

$$\tilde{a}\tilde{b} = \Big(\sum_{j=1}^{M_0 m-1} \tilde{\alpha}_{L_0 m+j} \tilde{P}_{L_0 m+j}\Big)\Big(\sum_{j=1}^{M_0 m-1} \tilde{P}_{j+L_0 m} b' \tilde{P}_j\Big)$$

$$= \sum_{j=1}^{M_0 m-1} \tilde{\alpha}_{L_0 m+j} \tilde{P}_{j+L_0 m} b' \tilde{P}_j = \omega \sum_{j=1}^{M_0 m-1} \tilde{\alpha}_j \tilde{P}_{j+L_0 m} b' \tilde{P}_j$$

$$= \omega\Big(\sum_{j=1}^{M_0 m-1} \tilde{P}_{j+L_0 m} b' \tilde{P}_j\Big)\Big(\sum_{j=1}^{M_0 m-1} \tilde{\alpha}_j \tilde{P}_j\Big) = \omega \tilde{b}\tilde{a}.$$

which is equivalent to $q(\tilde{a}, \tilde{b}) = 0$. Moreoever, $\tilde{a}$ and $\tilde{b}$ can be chosen as in the statement of the theorem by appropriate choice of m and n. Let $\epsilon = \|ab - \omega ba\|^{\frac{1}{p+2}}$. Since we have already established the estimate, $\|a-\tilde{a}\| \leq \|f-g\|_\infty = \frac{\pi}{M_0 m}$, just choose $m = \lceil \frac{M_0}{\pi\epsilon} \rceil$. Then we clearly have that $\|a-\tilde{a}\| \leq \epsilon$. Now, as in the proof presented in the previous chapter, by minimizing the right hand side of the expression

$$|||b-\tilde{b}||| \leq \tfrac{C_0 M_0 mn}{2\pi} |||ab-\omega ba||| + 8n^{-1/p}$$

with respect to n, gives us our estimate,

$$||a-\tilde{a}|| + |||b-\tilde{b}||| \leq K_{p,\omega} |||ab-\omega ba|||^{1/(p+2)}.$$

Hence it remains to prove the lemma.

Proof of Lemma 5.5.1:

Proof. To establish the equation, $b' = \tilde{b} + c_i$, we note that $b' - \tilde{b}$ is the sum of the terms of the form $P_s b' P_r$ so that $|s - L_0 mn - r| \leq 1$, and $s \in F_k, r \in F_j$ for $k \neq j + L_0 m$. This is only possible in two cases:

1. $r \in F_j, s = L_0 mn + r + 1 \in F_{j+L_0 m+1}$ and this happens when the index s is the left endpoint of $F_{j+L_0 m+1}$, and r is the right endpoint of F_j. In particular, $r = (j+1)n + i - 1$, and $s = (j + L_0 m + 1)n + i$. In this case, the term $P_s b' P_r$ is accounted for in c_i^u. For all other pairs, (s,r) coming from the intervals $F_{j+L_0 m+1}$ and F_j respectively, we have $|s-r| > L_0 mn + 1$.

2. $r \in F_j, s = L_0 mn + r - 1 \in F_{j+L_0 m-1}$. Then $P_s b' P_r$ is accounted for in c_i^ℓ.

□

Chapter 6

Self-Adjoint Operators that Almost Commute up to a Factor

6.1 Main Result

In this chapter we prove the following result:

Theorem 6.1.1. *Suppose* $(\mathcal{M}, \tau)$ *is a tracial von Neumann algebra and let* $a \in \mathcal{M}_{sa}$ *equipped with the operator norm* $||.||$ *on* $\mathcal{M}$ *and let* $b \in \mathcal{E}(\mathcal{M}, \tau)$*, where* $\|\!|.|\!\|$ *is the norm inherited from* $(\mathcal{E}, \|\!|.|\!\|)$*. Now, let* $\omega \in (-1, 1)$ *and set* $\kappa = \|\!|ab - \omega ba|\!\|$*. Then, for any* $\epsilon > 0$ *the operator* a *has a spectral projection* Q *so that*

$$\begin{aligned} &(i) \quad ||Qa|| \leq \epsilon \\ &(ii) \quad \|\!|b - QbQ|\!\| \leq K(\omega)\epsilon^{-1}\kappa, \end{aligned} \tag{6.1}$$

Here $K(\omega)$ *is a constant that depends only on* ω*.*

Remark 7. *Once this is established, we have that the self-adjoint pair of operators* (a, b) *is close to a pair of operators* $(\tilde{a}, \tilde{b})$ *that "trivially" commute up to a factor in the sense that if we take* $\tilde{a} = Q^{\perp}a$ *and* $\tilde{b} = QbQ$*, we have* $\|a - \tilde{a}\| \leq \epsilon$*,* $\|b - \tilde{b}\| \leq K(\omega)\epsilon^{-1}\kappa$*, and* $\tilde{a}\tilde{b} = \tilde{b}\tilde{a} = 0$*. Hence, in particular,* $\tilde{a}\tilde{b} = \omega\tilde{b}\tilde{a}$*.*

Remark 8. *Unlike the previous sections, we do not impose any special restriction on the symmetric function space involved. In particular, we can take* $E = L_{\infty}$*, then the corresponding non-commutative symmetric space will be our von Neumann algebra* $\mathcal{M}$*, with its operator norm.*

As a result of **Theorem 6.1.1.**, we get the following corollary,

Corollary 6.1.1. *Suppose* $a = a^*$*,* $b = b^*$*, and* $\omega \in (-1, 1)$*. Then for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *so that, whenever* $||a|| \leq 1$*, and* $\|ab - \omega ba\| < \delta$*, there exists* b' *and* a' *so that* $\|b - b'\| < \epsilon$ *and* $||a - a'|| < \epsilon$*. Moreover,* $a'b' = b'a' = 0$*.*

Proof. Let Q as in **Theorem 6.1.1**. Then we have that $a = QaQ + Q^{\perp}aQ^{\perp}$. So since we know $a' = Q^{\perp}aQ^{\perp}$, by functional calculus, $\|a - a'\| \leq \epsilon$ as well and furthermore $a'b' = b'a' = 0$.

□

6.2 Motivation for our Result

In this chapter we look at the same operator equation as the last chapter with the constraint that the operators a, b be self-adjoint. Formally, we consider:

Equation 6. $ab = \omega ba$*, with the restriction that* $a = a^*$ *and* $b = b^*$ $a, b \in \mathcal{B}(\mathcal{H})$*.*

It turns out that **Equation 6** is quite restrictive in the sense that if $\omega \in \mathbb{C} - \{-1, +1\}$, and $a, b \in B(\mathcal{H})$ satisfy **Equation 6**, then $ab = ba = 0$. Formally, we have:

Proposition 6.2.1. *Suppose $\omega \in \mathbb{C} - \{-1, 1\}$, and $a, b \in B(\mathcal{H})$ satisfy* **Equation 6**. *Then there exists a projection P so that $a = PaP$ and $b = P^{\perp}bP^{\perp}$ and consequently, $ab = ba = 0$.*

This proposition clearly fails for $\omega = \pm 1$, even when both a and b are assumed to be self-adjoint. Counterexamples are witnessed by any pair of commuting self-adjoint operators (for $\omega = 1$), or by anticommuting Pauli matrices ($\omega = -1$; such matrices are self-adjoint unitaries) which can also be found in [5]. Also, the Proposition fails if a and b are not self-adjoint, but merely normal. Indeed, denote by (e_k) the canonical basis in $\ell_2(\mathbb{Z})$. For $|\omega| = 1$, define a and b via

$$ae_k = e_{k+1}, \ be_k = \omega^{-k} e_k \ (k \in \mathbb{Z}).$$

Clearly, $ab = \omega ba$. For $|\omega| > 1$, this construction can be modified, with a being a one-sided (as opposed to bilateral) shift (hence a is no longer normal; however, b is self-adjoint when $\omega \in \mathbb{R}$).

In our proof below, we use the following result (Theorem 1.1 in [5]).

Theorem: Let A, B be bounded operators such that $AB \neq 0$ and $AB = \omega BA$, where $\omega \in \mathbb{C}$. Then:

1. if A or B is self-adjoint then $\omega \in \mathbb{R}$;
2. if both A and B are self-adjoint then $\omega \in \{-1, +1\}$;
3. if A and B are self-adjoint and one of them is positive then $\omega = 1$.

Proof of Proposition 6.2.1.

Proof. So suppose that $a = a^*, b = b^*$, and $ab = \omega ba = 0$ for some $\omega \notin \{-1, +1\}$, then $ab = ba = 0$. Indeed by Theorem 1.1 of [5], $ab = 0$, then $ba = (ab)^* = 0$. Now let P be the spectral projection onto the closure of the range of the operator a. Then $P^\perp$ is the projection on the kernel of the operator a. Since P is a spectral projection of a, we have that $a = PaP$. Moreover, for any $x \in Range(P)$, $ax \neq 0$, therefore $Pb = 0$. Suppose not, then pick a vector y so that $Pby \neq 0$, then $aPby = aby \neq 0$, a contradiction. Moreover, we also have that $bP = (Pb)^* = 0$, so we conclude that $b = P^\perp b P^\perp$.

□

6.3 Preliminaries

We begin with a proposition that will be used to prove our main result.

Proposition 6.3.1. *Suppose $a = a^*$, $||a|| \leq 1$, $\omega \in (-1, 1)$, and $\delta \in (0, 1 - |\omega|)$. Let $P = \chi_{[-(|\omega|+\delta), |\omega|+\delta]}(a)$ be a spectral projection of a. Then for any operator b there exists a perturbation b' so that $\|b - b'\| \leq C_0 \delta^{-1} \|ab - \omega ba\|$, and $b' P^\perp = 0$ (here, C_0 is an absolute constant).*

Proof. We can assume that $a = \sum_{s=1}^{S} \alpha_s R_s$, with $\alpha_s \in [-1, 1]$, and R_s being mutually orthogonal projections (instead of a, we can consider $H(a)$, where H is an appropriate step function). Note that the spectral projections of $H(a)$ are spectral projections of the operator a as well. For $t \in \mathbb{R}$ let $g(t) = e^{\iota at} b e^{-\iota \omega at} - b$, and $b' = \delta \int_{\mathbb{R}} (g(t) + b) \widehat{F}(\delta t) dt$. Here,

F is the function defined in beginning of Section 3.2. The integral converges. Furthermore, $b' - b = \delta \int_{\mathbb{R}} g(t)\widehat{F}(\delta t)dt$. We have $g'(t) = \iota e^{\iota at}(ab - \omega ba)e^{-\iota\omega at}$, hence $\|g'(t)\| = \|ab - \omega ba\|$ for any t. Consequently, $\|b - b'\| \le C_0\delta^{-1}\|ab - \omega ba\|$.

Now suppose $\xi_s \in$ Range R_s, and $\xi_r \in$ Range R_r. Then

$$\langle b'\xi_s, \xi_r\rangle = \delta \int_{\mathbb{R}} \left\langle e^{\iota at} b e^{-\iota\omega at}\xi_s, \xi_r \right\rangle \hat{F}(\delta t)dt$$

$$= \delta \int_{\mathbb{R}} \left\langle b e^{-\iota at}\xi_s, e^{-\iota\omega at}\xi_r \right\rangle \hat{F}(\delta t)dt = \delta \int_{\mathbb{R}} \left\langle e^{-\iota\alpha_s t} b\xi_s, e^{-\iota\omega\alpha_r t}\xi_r \right\rangle \hat{F}(\delta t)dt$$

$$= \langle b\xi_s, \xi_r\rangle \int_{\mathbb{R}} \delta e^{\iota(\omega\alpha_r - \alpha_s)t}\hat{F}(\delta t)dt$$

$$= \langle b\xi_s, \xi_r\rangle F\left(\tfrac{\omega\alpha_r - \alpha_s}{\delta}\right).$$

If $|\alpha_s| > |\omega| + \delta$, then $F((\omega\alpha_r - \alpha_s)/\delta) = 0$. Thus, $b'\xi_s = 0$. If $P = \chi_{[-(|\omega|+\delta),|\omega|+\delta]}(a)$, then Range $P^\perp = \vee_{|\alpha_s|>|\omega|+\delta}$ Range R_s. Thus, $b'P^\perp = 0$.

□

Corollary 6.3.1. *Suppose $a = a^*$, $b = b^*$, $||a|| \le 1$, $\omega \in (-1, 1)$, and $\delta \in (0, 1 - |\omega|)$. Consider the spectral projection $P_1 = \chi_{[-(|\omega|+\delta),|\omega|+\delta]}(a)$, and let $a_1 = P_1 a P_1 = aP_1 = P_1 a$, and $b_1 = P_1 b P_1$. Then*

$$(6.2) \qquad \begin{aligned} &(i) \quad ab_1 - \omega b_1 a = a_1 b_1 - \omega b_1 a_1 = P_1(ab - \omega ba)P_1 \\ &(ii) \quad \|a_1 b_1 - \omega b_1 a_1\| \le \|ab - \omega ba\| \\ &(iii) \quad \|b_1 - b\| \le c\delta^{-1}\|ab - \omega ba\| \end{aligned}$$

where c is a universal constant.

Proof. First we note that the operator b' comes from Proposition 6.3.1. Now, we begin by proving (iii) first: Let $b_1 = P_1 b P_1$, then $b_1 = b_1^*$. We have $bP_1^\perp = (b - b')P_1^\perp$, hence $\|b - bP_1\| = \|bP_1^\perp\| = \|(b - b')P_1^\perp\| \leq \|b - b'\|$.

Furthermore, $\|b - b_1\| = \|(b - bP_1) + (b - bP_1)^* P_1\| \leq 2\|b - bP_1\| \leq 2C_0\delta^{-1}\|ab - \omega ba\|$ which yields (iii).

For (i), we have that $\|ab_1 - \omega b_1 a\| \leq \|ab - \omega ba\|$. Indeed,

$$a = P_1 a P_1 + P_1^\perp a P_1^\perp$$

hence

$$ab - \omega ba = (P_1 a P_1 b - \omega b P_1 a P_1) + (P_1^\perp a P_1^\perp b - \omega b P_1^\perp a P_1^\perp).$$

Furthermore,

$$ab_1 - \omega b_1 a = P_1 a P_1 b P_1 - \omega P_1 b P_1 a P_1 = P_1(ab - \omega ba)P_1.$$

So (ii) follows from the fact that P_1 is a contraction.

□

Corollary 6.3.2. *Suppose* $a = a^*$, $b = b^*$, $\omega \in (-1, 1)$, $\delta \in (0, 1 - |\omega|)$, *and* $C \geq ||a||$. *Then* a *has a spectral projection* R *so that, for* $a_1 = RaR = aR = Ra$, *and* $b_1 = RbR$, *we have:*

$$
(6.3) \qquad
\begin{array}{ll}
(i) & ||Ra|| \leq (|\omega| + \delta)C \\
(ii) & ab_1 - \omega b_1 a = a_1 b_1 - \omega b_1 a_1 = R(ab - \omega ba)R \\
(iii) & \|a_1 b_1 - \omega b_1 a_1\| \leq \|ab - \omega ba\| \\
(iv) & \|b_1 - b\| \leq c\delta^{-1} \|ab - \omega ba\|
\end{array}
$$

Sketch of a proof. Apply Corollary 6.3.1.to the operators $a_0 = a/C$ and b. □

6.4 Proof of Main Result

Proof of Theorem 6.1.1.

Pick $\delta \in (0, 1 - |\omega|)$ and let $\alpha = \frac{1}{|\omega| + \delta} > 1$.

We now use an iterative procedure by repeated applications of **Corollary 6.3.2**:

Starting conditions: Let $a_0 = a, b_0 = b, C_0 = ||a_0||, \kappa = \|a_0 b_0 - \omega b_0 a_0\|$

Iteration 1: Apply **Corollary 6.3.2** to our initial data to obtain spectral projection P_1 of a and operators a_1 and b_1 such that:

$a_1 = P_1 a P_1 = a P_1 = P_1 a$, $b_1 = P_1 b P_1$

$||a_1|| \leq C_1 = C_0/\alpha, \|b_1 - b_0\| \leq c\delta^{-1}\kappa/C_0,$

$\|a_1 b_1 - \omega b_1 a_1\| \leq \kappa$

Iteration at step k: Apply **Corollary 6.3.2** to the operators in the previous iteration to obtain spectral projection P_k of a and operators a_k and b_k such that:

$a_k = P_k a_{k-1} P_k = a_{k-1} P_k = P_k a_{k-1}, b_k = P_k b_{k-1} P_k$

$||a_k|| \leq C_k = C_0/\alpha^k$, $|||b_k - b_{k-1}||| \leq c\delta^{-1}\kappa/C_{k-1} \leq c\delta^{-1}\alpha^{k-1}\kappa/C_0$, and $|||a_k b_k - \omega b_k a_k||| \leq \kappa$.

End of Iteration: Let us stop after m steps where m is the least integer so that $||a_0||\alpha^{-m} \leq \epsilon$. Then we get $||a_m|| \leq \epsilon$, and $||a_0||/\epsilon \leq \alpha^m < ||a_0||\alpha/\epsilon$.

Moreover,

$$|||b_m - b_0||| \leq \sum_{k=1}^m |||b_k - b_{k-1}||| \leq c\delta^{-1}||a||^{-1}\kappa \sum_{k=1}^m \alpha^{k-1}$$

$$\leq c\delta^{-1}(\alpha^m - 1)(\alpha - 1)^{-1}\kappa||a||^{-1} \leq c\delta^{-1}\alpha^m(\alpha - 1)^{-1}\kappa||a||^{-1}.$$

However, $\alpha^m \leq ||a_0||\alpha/\epsilon$, so by setting $\tilde{b} := b_m$ we get that $|||\tilde{b} - b_0||| \leq K\epsilon^{-1}\kappa$, where $K = c\delta^{-1}(\alpha - 1)^{-1}\alpha$. Finally, note that by setting $\tilde{a} = a_m$, we get that $||\tilde{a}|| \leq \epsilon$ and by repeated applications of **Corollary 6.3.2** there exists a spectral projection $Q = P_m$ of a such that $a_m = Qa_m$. Hence, $||Qa|| \leq ||P_m a P_m|| = ||a_m|| \leq \epsilon$.

Remark 9. *In the proof above we use the fact that the operators P_k are in fact spectral projections of the operator a.*

For instance, P_2 is a spectral projection of the operator a. Recall that $P_1 = \chi_{[-c_1,c_1]}(a)$ for some $c_1 > 0$. Now define the functions:

$$f(t) = \begin{cases} t & |t| \in [-c_1, c_1] \\ 0 & t \notin [-c_1, c_1] \end{cases}$$

$$g(t) = \begin{cases} t & |t| \notin [-c_1, c_1] \\ 0 & t \in [-c_1, c_1] \end{cases}$$

Then $a_1 = f(a)$, and $a - a_1 = g(a)$. Furthermore, $P_2 = \chi_{[-c_2,c_2]}(a_1)$ for some $c_2 \leq c_1$. In other words, $P_2 = \chi_{[-c_2,c_2]}(f(a))$. By functional calculus, the right hand side equals $(\chi_{[-c_2,c_2]} \circ f)(a)$, where $\circ$ denotes the composition of functions. But $\chi_{[-c_2,c_2]} \circ f = \chi_{[-c_2,c_2]}$.

Chapter 7

Counter-examples

7.1 Voiculescu's Unitaries

The following $n \times n$ unitary matrices,

$$(7.1) \qquad S_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad T_n = \begin{pmatrix} c_n & 0 & 0 & \cdots & 0 \\ 0 & c_n^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_n^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & c_n^n \end{pmatrix}$$

are named after Dan Voiculescu who provided the first counter-example to **Problem 1** in [30]. Here $c_n = e^{2\pi i/n}$ is a root of unity. To be more specific, Voiculescu proved:

Theorem 7.1.1. *(Voiculescu 1983)Let $||.||$ denote the operator norm. Then we have that $\lim_{n\to\infty} ||[S_n, T_n]|| = 0$ but there do not exist unitary $\{S_n'\}$ and $\{T_n'\}$ such that $[S_n', T_n'] =$*

0 and $\lim_{n\to\infty}(||S_n - S_n'|| + ||T_n - T_n'||) = 0$.

In [12] Ruy Exel and Terry Loring use Voiculescu's Unitaries in a very elegant and elementary proof to establish:

Theorem 7.1.2. *There exist* $T_n, S_n \in M_n$ *such that* $\lim_{n\to\infty} ||[S_n, T_n]|| = 0$*, yet if* $X, Y \in M_n$ *commute, then* $\max\{||X - T_n||, ||Y - S_n||\} \geq \sqrt{2 - |1 - c_n|} - 1$*, where* $c_n = e^{2\pi i/n}$ *and* $||.||$ *is the usual operator norm.*

7.2 A Counter-Example with respect to the Schatten p-norms for $1 < p \leq \infty$.

In this section we use Voiculescu's Unitaries and Exel and Loring's method of proof, with modifications, to establish:

Theorem 7.2.1. *Let* $\|.\|_p$ *denote the Schatten p-norm. For* $1 < p \leq \infty$ *there exist unitaries* $T_n, S_n \in M_n(\mathbb{C})$ *such that* $\lim_{n\to\infty} \|[S_n, T_n]\|_p = 0$*, yet if* $X, Y \in M_n(\mathbb{C})$ *commute, then* $\max\{\|X - T_n\|_p, \|Y - S_n\|_p\} \geq \max\{||X - T_n||, ||Y - S_n||\} \geq \sqrt{2 - |1 - c_n|} - 1$*, where* $c_n = e^{2\pi i/n}$.

Proof. We borrow heavily on Exel and Loring's method of proof with some minor modifications. Once again, we make use of Voiculuescu's Unitaries S_n and T_n.

A simple computation shows that:

$$(7.2)\quad \sqrt{(S_nT_n - T_nS_n)^*(S_nT_n - T_nS_n)} = \begin{pmatrix} |1-c_n| & 0 & 0 & \cdots & 0 \\ 0 & |1-c_n| & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |1-c_n| & 0 \\ 0 & 0 & \cdots & 0 & |1-c_n| \end{pmatrix}$$

A few well known facts about S_n and T_n:

(a) $||T_nS_n - S_nT_n|| = |1 - c_n|$

(b) $\|T_nS_n - S_nT_n\|_p^p = n|1 - c_n|^p$

(c) $\det(T_n) = \det(S_n) = (-1)^{n+1}$

(d) $S_nT_nS_n^* = \overline{c_n}T_n$

Note that for $p > 1$ we have that $\|T_nS_n - S_nT_n\|_p = n^{1/p}|1 - c_n| = 2n^{1/p}\sin(\pi/n) \to 0$ as $n \to \infty$. Now let $X, Y \in M_n$ be commuting matrices and let $d = \max\{\|X - T_n\|_p, \|Y - S_n\|_p\}$. Now note that $d \geq \max\{||X - S_n||, ||Y - T_n||\}$ since the Schatten p-norm of a matrix is always greater than the operator norm of the same matrix. Hence, we can just invoke Exel and Loring's theorem to deduce our result:

$$\max\{\|X - T_n\|_p, \|Y - S_n\|_p\} \geq \sqrt{2 - |1 - c_n|} - 1.$$

□

7.3 A Counter-example on Unitary Matrices that Almost Commute up to a Factor.

Recall that in chapter 5 we looked at the operator equation:

$$ab = \omega ba \text{ where } \omega \in \mathbb{T}$$

and its *approximate* version. Assuming that the operator a is unitary, we were able to show that operators, a and b which almost commute up to a factor with respect to the normalized Schatten p-norm are near operators $\tilde{a}$ and $\tilde{b}$ that commute up to the same factor and approximate the operators a and b. We have discovered that in the matricial setting, Exel and Loring method of proof, with modifications, also yields the following result:

Theorem 7.3.1. *Let $\omega_n = e^{2\pi ik/n} \in \mathbb{T}$ and fix $\epsilon > 0$. Then there is a natural number N such that if $m \geq N$ has the same parity as n then there exist unitary matrices $A, B \in M_{mn\times mn}$ such that $||AB - \omega_n BA|| < \epsilon$, yet if $XY = \omega_n YX$, then $\max\{||X - A||, ||Y - B||\} > C > 0$ for some universal constant C.*

Proof. In this proof we will use the Kronecker product of two matrices and its relation to other matrix operations.

Definition 7.3.1. *If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:*

$$A \otimes B = \begin{pmatrix} a_{1,1}B \cdots & a_{1,n}B \\ \vdots & \vdots \\ a_{m,1}B \cdots & a_{m,n}B \end{pmatrix}$$

For our purposes, will use the following properties of the Kronecker product:

(a) $A \otimes (B + C) = A \otimes B + A \otimes C$

(b) $(A + B) \otimes C = A \otimes C + B \otimes C$

(c) $(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$

(d) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

(e) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

(f) $(A \otimes B)^* = (A^* \otimes B^*)$

(g) If $A \in M_n(\mathbb{C}), B \in M_p(\mathbb{C})$ then $\det(A \otimes B) = (\det A)^p (\det B)^n$.

Now we continue on as before, borrowing heavily from Excel and Loring's proof with modifications. Fix $\epsilon > 0$. Now, choose N large enough so that for $m \geq N$, $|1 - c_m| < \epsilon$ where $c_m = e^{2\pi i/m}$. Assume that m has the same parity as n. Set $A = S_m \otimes T_n'$ and $B = T_m \otimes S_n$ where S_n, T_n are Voiculescu's unitaries of size n, while S_m, T_m are Voiculescu's unitaries of size m. Now let $T_n' = diag(\omega_n, \omega_n^2, ..., \omega_n^n)$. That is, $A, B \in M_{mn \times mn}$ have the following explicit forms:

$$A = S_m \otimes T'_n = \begin{pmatrix} \omega_n S_m & 0 & 0 & \cdots & 0 \\ 0 & \omega_n^2 S_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_n^{n-1} S_m & 0 \\ 0 & 0 & \cdots & 0 & \omega_n^n S_m \end{pmatrix}$$

and

$$B = T_m \otimes S_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & T_m \\ T_m & 0 & 0 & \cdots & 0 \\ \vdots & T_m & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & T_m & 0 \end{pmatrix}$$

So that

$$AB - \omega_n BA =$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & \omega_n S_m T_m - \omega_n T_m S_m \\ \omega_n^2 S_m T_m - \omega_n^2 T_m S_m & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \omega_n^n S_m T_m - \omega_n^n T_m S_m & 0 \end{pmatrix}$$

Then $||AB - \omega_n BA|| = ||S_m T_m - T_m S_m|| = |1 - c_m| < \epsilon$. Now suppose that $X, Y \in M_{mn\times mn}$ such that $XY = \omega_n YX$ and set

$$d = \max\{||X - A||, ||Y - B||\}$$

For $t \in [0,1]$ let $A_t = A + t(X - A), B_t = B + t(Y - B)$ and let

$$\gamma_t(r) = \det((1-r)A_t B_t + r\omega_n B_t A_t) \text{ for } r \in [0,1]$$

Note that

$$\det(A) = \det(S_m \otimes T_n') = \det(S_m)^n \det(T_n')^m$$

$$= (-1)^{(m+1)n}(\det(T_n'))^m = (-1)^{mn+n}(\omega_n^{1+\ldots+n-1})^m$$

$$= (-1)^{mn+n}(\omega_n^{1+\ldots+n-1})^m = (-1)^{mn+n}(\omega_n^{n(n-1)/2})^m$$

$$= (-1)^{mn+n}(-1)^{mk(n-1)} = (-1)^{n(m+1)+mk(n-1)} = 1\ .$$

if m and n have the same parity. Now,

$$\det(B) = \det(T_m \otimes S_n) = \det(T_m)^n \det(S_n)^m$$

$$= (-1)^{n(m+1)}(-1)^{mn+1} = (-1)^{m+n} = 1$$

if m and n have the same parity. For $t = 1$, $A_t = X$ and $B_t = Y$ which commute up to the factor ω_n, so $\gamma_1(r) = \det(XY)$ for all $r \in [0,1]$ which is just a constant curve. When $t = 0$, however, we have that $A_t = A$ and $B_t = B$. Now we make a computation to help us compute a determinant that we will use shortly. To this end we use the properties of the Kronecker product outlined in the beginning of the proof.

In particular, we have that,

$$BAB^* = (T_m \otimes S_n)(S_m \otimes T'_n)(T^*_m \otimes S^*_n)$$

$$= (T_m S_m T^*_m \otimes S_n T'_n S^*_n) = (c_m S_m \otimes S_n T'_n S^*_n)$$

$$= (c_m S_m \otimes \omega_n^{n-1} T'_n) = c_m \omega_n^{n-1}(S_m \otimes T'_n)$$

$$= c_m \omega_n^{n-1} A$$

Hence,

$$\gamma_0(r) = \det((1-r)AB + r\omega_n BA) = \det((1-r)A + r\omega_n BAB^*)\det(B)$$

$$= \det((1-r)A + r\omega_n c_m \omega_n^{(n-1)} A)\det(B) = (1 - r + rc_m)^{mn}\det(B)\det(A)$$

$$= (1 - r + rc_m)^{mn}$$

when m and n have the same parity. As the parameter r goes from 0 to 1, $(1-r+rc_m)$ moves along the line segment joining 1 to $c_m = e^{2\pi i/n}$. It follows that $\gamma_0(r)$ is never zero and that it winds around the origin counterclockwise at least once. Now, since the winding number is a homotopy invariant of closed curves in the complex plane excluding the origin, we will get a contradiction as soon as we prove that $\gamma_t(r)$ is never zero. Hence, it suffices to show that matrices $(1-r)A_tB_t + r\omega_n B_tA_t$ are invertible for all $t, r \in [0,1]$. We will accomplish this by showing that the latter matrix is at a distance less than one from the unitary matrix AB.

We have,

$$||(1-r)A_tB_t + r\omega_n B_tA_t - AB|| \leq (1-r)||A_tB_t - AB|| + r||\omega_n B_tA_t - AB||$$

$$\leq (1-r)(||A_tB_t - A_tB|| + ||A_tB - AB||)+$$

$$r(||\omega_n B_tA_t - \omega_n BA_t|| + ||\omega_n BA_t - \omega_n BA|| + ||\omega_n BA - AB||)$$

$$\leq (1-r)(||A_t||||B_t - B|| + ||A_t - A||||B||)+$$

$$r(||B_t - B||||A_t|| + ||B||||A_t - A|| + |1-c_m|)$$

$$\leq (1-r)((1+d)d + d) + r(d(1+d) + d + |1-c_m|)$$

$$= (1+d)d + d + r|1-c_m| \leq d^2 + 2d + |1-c_m|.$$

Now, if $d < \sqrt{2 - |1-c_m|} - 1$ we have that $d^2 + 2d + |1-c_m| < 1$. A contradiction.

□

Chapter 8

Normal Completions

8.1 Introduction

In [1] R. Bhatia and M. Choi ask the following question: What matrix pairs (B, C) can be the off-diagonal entries of a $2n \times 2n$ normal matrix N of the form

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The Matrix N is said to be a "Normal Completion" for the matrix pair $(B, C) \in M_n(\mathbb{C}) \times M_n(\mathbb{C})$. We begin our investigation of this problem by studying the normality condition $NN^* = N^*N$ which gives us a system of matrix equations:

(1) $AA^* - A^*A = C^*C - BB^*$

(2) $DD^* - D^*D = B^*B - CC^*$

(3) $AC^* - C^*D = A^*B - BD^*$

This reduces the problem to finding matrices $A, D \in M_n(\mathbb{C})$ satisfying these three equations. We do not know of any general algorithm for solving such systems of equations and for many pairs of matrices (B, C) we cannot even determine whether or not a solution exists.

8.2 Results

Theorem 8.2.1. *If B and C are commuting normal matrices, then (B, C) admits a normal completion with $A = D = O$, the $n \times n$ zero matrix.*

Proof. Since B and C are normal and commute, we may simultaneously diagonalize them. That is, there exists an unitary matrix P such that P^*CP and P^*BP are diagonal matrices. So we may assume that:

$$C = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 \\ 0 & c_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n,n} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{1,1} & 0 & \cdots & 0 \\ 0 & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n,n} \end{pmatrix}$$

Now we want to find $n \times n$ matrices A and D such that the following matrix:

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is normal.

When we choose A and D both to be the $n \times n$ zero matrix we get:

$$NN^* = \begin{pmatrix} |b_{1,1}|^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & |b_{2,2}|^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |b_{n,n}|^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & |c_{1,1}|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & |c_{2,2}|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & |c_{n,n}|^2 \end{pmatrix}$$

and

$$N^*N = \begin{pmatrix} |c_{1,1}|^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & |c_{2,2}|^2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |c_{n,n}|^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & |b_{1,1}|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & |b_{2,2}|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & |b_{n,n}|^2 \end{pmatrix}$$

Hence, since $BC = CB$, we see that in this case that the matrix N is normal since $|b_{i,i}| = |c_{i.i}|$ for all $1 \leq i \leq n$.

□

Theorem 8.2.2. *If $B = diag(B_1, B_2)$ and $C = diag(C_1, C_2)$ are $2n \times 2n$ block-diagonal self-adjoint matrices with real entries that satisfy the following system of matrix equations:*

$$(8.1) \qquad \begin{aligned} (i) &\qquad B_1^2 + B_2^2 = C_1^2 + C_2^2 \\ (ii) &\quad B_1C_1 + B_2C_2 = C_1B_1 + C_2B_2 \end{aligned}$$

then matrix pair (B, C) admits a $4n \times 4n$ normal completion.

Proof. Let

$$N = \begin{bmatrix} 0 & B_2 & B_1 & 0 \\ C_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & B_1 \\ 0 & C_2 & C_1 & 0 \end{bmatrix}$$

where $B_j, C_j \in M_n(\mathbb{R})$. Note that if we view this matrix as a part of a chessboard then the configuration or placement of the matrices B_j, C_j are a "knights move" away from themselves. By comparing the entries in the normality equation, $NN^* = N^*N$, we see that (B, C) admits a normal completion if equations (i) and (ii) are satisfied.

□

Remark 10. *One set of solutions to the matrix equations in the previous theorem can be constructed from solutions of the quartic* $V(a^2 + b^2 = c^2 + d^2)$ *whose solutions can be parameterized by* $(a, b, c, d) = (pr + qs, qr - ps, pr - qs, ps + qr)$ *where* p, q, r, s *are arbitrary real numbers. A simple calculation, then, shows that if we choose* $B_1 = diag(a, ..., a), B_2 = diag(b, ..., b), C_1 = diag(c, ..., c), C_2 = diag(d, ..., d)$ *then the set of matrix equations are satisfied and hence we have constructed an explicit class of matrices that admit normal completions.*

8.3 Projections of the same rank

Theorem 8.3.1. *Suppose that $B, C \in M_n(\mathbb{C})$ are projections of the same rank, then the matrix pair (B, C) admits a normal completion.*

Proof. Recall that if $a, b \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space, then we may define an operator from $\mathcal{H} \bigoplus \mathcal{H} \to \mathcal{H}$ by setting $(a \otimes b)(h) = \langle h, b \rangle a$.

Some properties of these operators that we will use are:

(1) $(a \otimes b)^* = (b \otimes a)$

(2) $(a \otimes b)(c \otimes d) = \langle c, b \rangle (a \otimes d)$

Let $H = Range(B)$ and $K = Range(C)$. By assumption, $r = \dim(H) = \dim(K)$. We find orthonormal basis $(\xi_j)_{j=1}^r$, and $(\zeta_j)_{j=1}^r$ in K and H respectively, so that:

(1) $\langle \xi_i, \zeta_i \rangle$ is real for all $1 \leq i \leq r$, and

(2) $\langle \xi_i, \zeta_j \rangle = 0$ for $i \neq j$.

Assume, for the sake of simplicity that $B|_K$ has trivial kernel. We may assume further, that (ξ_j) is the orthonormal basis corresponding to the singular values of $B|_K$, that is, $B\xi_j = c_j \zeta_j$ where $c_j = ||B\xi_j||$. Now, for $i \neq j$ we claim that $\langle \xi_i, \zeta_j \rangle = 0$. Note that $\xi_j = c_j \zeta_j + \eta_j$ where $\eta_j = B^{\perp} \zeta_j \perp H$. Then, we have that $\langle \xi_i, \zeta_j \rangle = \langle c_i \zeta_i + \eta_i, \zeta_j \rangle = 0$, as claimed. Now we can write $B = \sum_{j=1}^r \zeta_j \otimes \zeta_j$ and $C = \sum_{j=1}^r \xi_j \otimes \xi_j$. By first looking at the case where $r = 1$, we found that good candidates for A and D are $\sum_{j=1}^r \xi_j \otimes \zeta_j$ and $\sum_{j=1}^r \zeta_j \otimes \xi_j$, respectively.

Recall that for matrices of the type:

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the normality condition, $NN^* = N^*N$ gives us a system of matrix equations which we will state again for the sake of the reader:

(1) $AA^* - A^*A = C^*C - BB^*$

(2) $DD^* - D^*D = B^*B - CC^*$

(3) $AC^* - C^*D = A^*B - BD^*$

Hence, we will be done once we show that our matrices, $A, B, C, D \in M_n$, satisfy (1), (2), and (3).

To establish (1), note that

$$AA^* = (\textstyle\sum_{j=1}^{r} \xi_j \otimes \zeta_j)(\sum_{j=1}^{r} \zeta_j \otimes \xi_j)$$

$$= \textstyle\sum_{j=1}^{r} \langle \zeta_j, \zeta_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j} (\xi_i \otimes \zeta_i)(\zeta_j \otimes \xi_j)$$

$$= \textstyle\sum_{j=1}^{r} (\xi_j \otimes \xi_j) + \sum_{i \neq j} \langle \zeta_j, \zeta_i \rangle (\xi_i \otimes \xi_j)$$

$$= \textstyle\sum_{j=1}^{r} (\xi_j \otimes \xi_j) = C^*C$$

Similarly,

$$A^*A = (\textstyle\sum_{j=1}^{r} \xi_j \otimes \zeta_j)^* (\sum_{j=1}^{r} \xi_j \otimes \zeta_j)$$

$$= (\textstyle\sum_{j=1}^{r} \zeta_j \otimes \xi_j)(\sum_{j=1}^{r} \xi_j \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}(\zeta_j \otimes \zeta_j) + \sum_{i \neq j}\langle \zeta_j, \zeta_i \rangle (\xi_i \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}(\zeta_j \otimes \zeta_j) = BB^*$$

which shows that (1) is satisfied. Moreover,

$$DD^* = (\sum_{j=1}^{r} \zeta_j \otimes \xi_j)(\sum_{j=1}^{r} \xi_j \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}\langle \xi_j, \xi_j \rangle (\zeta_j \otimes \zeta_j) + \sum_{i \neq j}(\zeta_i \otimes \xi_i)(\xi_j \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}(\zeta_j \otimes \zeta_j) + \sum_{i \neq j}\langle \xi_j, \xi_i \rangle (\zeta_i \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}(\zeta_j \otimes \zeta_j) = B^*B$$

and

$$D^*D = (\sum_{j=1}^{r} \xi_j \otimes \zeta_j)(\sum_{j=1}^{r} \zeta_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r}\langle \zeta_j, \zeta_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j}(\xi_i \otimes \zeta_i)(\zeta_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r}(\xi_j \otimes \xi_j) + \sum_{i \neq j}\langle \xi_j, \xi_i \rangle (\zeta_i \otimes \zeta_j)$$

$$= \sum_{j=1}^{r}(\xi_j \otimes \xi_j) = C^*C$$

which shows that equation (2) is satisfied. We also have that

$$AC^* = (\sum_{j=1}^{r} \xi_j \otimes \zeta_j)(\sum_{j=1}^{r} \xi_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r}\langle \xi_j, \zeta_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j}(\xi_i \otimes \zeta_i)(\xi_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r}\langle \xi_j, \zeta_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j}\langle \xi_j, \zeta_i \rangle (\xi_i \otimes \xi_j)$$

$$= \sum_{j=1}^{r} \langle \xi_j, \zeta_j \rangle (\xi_j \otimes \xi_j)$$

and

$$C^* D = (\sum_{j=1}^{r} \xi_j \otimes \xi_j)(\sum_{j=1}^{r} \zeta_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r} \langle \zeta_j, \xi_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j} (\xi_i \otimes \xi_i)(\zeta_j \otimes \xi_j)$$

$$= \sum_{j=1}^{r} \langle \zeta_j, \xi_j \rangle (\xi_j \otimes \xi_j) + \sum_{i \neq j} \langle \zeta_j, \xi_i \rangle (\xi_i \otimes \xi_j)$$

$$= \sum_{j=1}^{r} \langle \zeta_j, \xi_j \rangle (\xi_j \otimes \xi_j)$$

which shows that $AC^* - C^*D = O$. A similar calculation shows that we also have that $A^*B - BD^* = O$. Hence, equation (3) is also satisfied and so our matrix:

$$N = \begin{bmatrix} \sum_{j=1}^{r} (\xi_j \otimes \zeta_j) & \sum_{j=1}^{r} (\zeta_j \otimes \zeta_j) \\ \sum_{j=1}^{r} (\xi_j \otimes \xi_j) & \sum_{j=1}^{r} (\zeta_j \otimes \xi_j) \end{bmatrix}$$

is normal.

$\square$

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