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UNIFORM LOCALIZATION IS ALWAYS UNIFORM

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ABSTRACT. In this note we show that if a family of ergodic Schrödinger operators on $l^2(\mathbb{Z}^\gamma)$ with continuous potentials have uniformly localized eigenfunctions, then these eigenfunctions must be uniformly localized in a homogeneous sense.

1. Introduction

Given a topological space Ω, let $T_i : \Omega \to \Omega$ be commuting homeomorphisms, and let μ be an ergodic Borel measure on Ω. Let $f : \Omega \to \mathbb{R}$ be continuous and define $V_\omega(n) = f(T^n\omega)$ for $n \in \mathbb{Z}^\gamma$, where $T^n = T_1^{n_1} \dots T_\gamma^{n_\gamma}$. Let H_ω be the operator on $l^2(\mathbb{Z}^\gamma)$,

$$(H_\omega u)(n) = \sum_{|m-n|=1} u(m) + V_\omega(n)u(n).$$

The occurrence of pure point spectrum for the operators $\{H_\omega\}$ is called *phase stable* if it holds for every $\omega \in \Omega$.

For a self-adjoint operator H on $l^2(\mathbb{Z}^\gamma)$, we say that H has uniformly localized eigenfunctions (ULE), if H has a complete set of orthonormal eigenfunctions $\{\phi_n\}_{n=1}^\infty$, and there are $\alpha > 0$, $C > 0$, such that

$$|\phi_n(m)| \leq Ce^{-\alpha|m-m_n|}$$

for all eigenfunctions ϕ_n and suitable m_n. It is known that ULE has a close connection with phase stability of pure point spectrum. Actually, in paper [3], Jitomirskaya pointed out that instability of pure point spectrum implies absence of uniform localization. It is also shown in [5] that if H_ω has ULE for ω in a set of positive μ-measure, then H_ω has pure point spectrum for any $\omega \in supp(\mu)$. The proof of this statement mainly relies on the fact that ULE implies uniform dynamical localization (UDL), which means if H_ω has ULE, then

$$|(\delta_l, e^{-itH_\omega}\delta_m)| \leq C_\omega e^{-\alpha_\omega|l-m|}$$

for some constants α_ω, and C_ω that depend on ω. Recently, in [1] and [2], Damanik and Gan established ULE for a certain model and then proved that for this model, actually C_ω and α_ω can be chosen to be independent of ω. In this note we will show that the latter property is always a corollary of ULE.

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First, let us give a new definition.

Definition 1.1. H_ω has uniform or homogeneous ULE in a set S means that H_ω has ULE for any ω in S and

$$|\phi_n^\omega(m)| \leq Ce^{-\alpha|m-m_n^\omega|}$$

with constants $\alpha > 0$ and $C > 0$ which do not depend on ω.

Then the main theorems in the note can be stated as follows:

Theorem 1.2. *If H_ω has ULE for ω in a positive μ-measure set, then H_ω has homogeneous ULE in supp(μ).*

Theorem 1.2′. *If T is minimal, and H_ω has ULE at a single ω, then H_ω has homogenous ULE in Ω.*

2. Proof of Theorem 1.2

Let $\{\mathbf{U}_k\}$ be a family of transitions on $l^2(\mathbb{Z}^\gamma)$ defined by $(\mathbf{U}_k u)(m) := u(m-k)$. Clearly, if $\{\phi_n^\omega\}$ is a complete set of eigenfunctions of H_ω, then $\{\mathbf{U}_k\phi_n^\omega\}$ is a complete set of eigenfunctions of $H_{T^k\omega}$. Also, if H_ω has ULE, which means $\exists\ \alpha_0 > 0, C_0 > 0$ such that

$$|\phi_n^\omega(m)| \leq C_0 e^{-\alpha_0|m-m_n^\omega|}$$

for all eigenfunctions ϕ_n^ω and suitable m_n^ω, then $H_{T^k\omega}$ also has ULE. In fact if we let $m_n^{T^k\omega} = m_n^\omega + k$, then

$$|\phi_n^{T^k\omega}(m)| = |(\mathbf{U}_k\phi_n^\omega)(m)| \leq C_0 e^{-\alpha_0|m-m_n^{T^k\omega}|}$$

for all eigenfunctions $\mathbf{U}_k\phi_n^\omega$. Also notice that the constants C_0 and α_0 are the same for H_ω and $H_{T^k\omega}$.

Lemma 2.1. *If H_ω has ULE for ω in a positive μ-measure set S, then H_ω has ULE for a.e. $\omega \in \Omega$.*

Proof. H_ω has ULE in $\bigcup_{k\in\mathbb{Z}^\gamma} T^k S$, which is a transition invariant set, so

$$\mu\left(\Omega\backslash\left(\bigcup_{k\in\mathbb{Z}^\gamma} T^k S\right)\right) = 0. \qquad \square$$

Theorem 2.2. *If H_ω has ULE for a.e. $\omega \in \Omega$, then $\exists\ \alpha > 0$ independent of ω, such that*

$$|\phi_n^\omega(m)| \leq C_\omega e^{-\alpha|m-m_n^\omega|}$$

for a.e. $\omega \in \Omega$ and all eigenfunctions ϕ_n^ω with suitable m_n^ω.

Proof. Let $\bigcup_{j=1}^\infty\{\omega|\ |\phi_n^\omega(m)| \leq C_\omega e^{-\frac{1}{j}|m-m_n^\omega|}$, for all eigenfunctions ϕ_n^ω and suitable $m_n^\omega\} := \bigcup_{j=1}^\infty A_j$. A_j is translation invariant. Since $\mu(\bigcup_{j=1}^\infty A_j) = 1$, $\exists\ j_0$ such that $\mu(A_{j_0}) = 1$. $\square$

Now, let's return to the proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.2 and direct computation we have

$$|(\delta_l, e^{-itH_\omega}\delta_m)| \leq C_\omega e^{-\alpha|l-m|}$$

for a.e. ω. Let

$$F(\omega) = \sup_{t\in\mathbb{Q},\ l,m\in\mathbb{Z}^\gamma} |(\delta_l, e^{-itH_\omega}\delta_m)|e^{\alpha|l-m|}.$$

Then $F(\omega) < \infty$ a.e. ω. It is also easy to see that $F(\omega)$ is measurable and translation invariant. Therefore, by ergodicity, $F(\omega) = C$ a.e. ω. Hence $|(\delta_l, e^{-itH_\omega}\delta_m)| \leq Ce^{-\alpha|l-m|}$ a.e. ω. Then on a dense set in $supp(\mu)$,

$$|(\delta_l, e^{-itH_\omega}\delta_m)| \leq Ce^{-\alpha|l-m|}.$$

By continuity, the inequality holds for any ω in $supp(\mu)$.

Then since $P^\omega_{\{E\}} = s - \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{iEs}e^{-iH_\omega s}ds$, we have

$$|(\delta_l, P^\omega_{\{E\}}\delta_m)| \leq Ce^{-\alpha|l-m|}$$

for any $E \in \mathbb{R}$. Therefore if we choose $\tilde{m}^\omega_n$ so that $|\phi^\omega_n(\tilde{m}^\omega_n)| = \sup_m |\phi^\omega_n(m)|$, we get

$$|\phi^\omega_n(l)|^2 \leq |\phi^\omega_n(l)||\phi^\omega_n(\tilde{m}_n)| \leq Ce^{-\alpha|l-\tilde{m}^\omega_n|}.$$

□

Remark 2.3. For the proof of Theorem 1.2′, one needs to realize that when T is minimal, $F(\omega)$ being translation invariant implies that $F(\omega)$ is constant in a dense subset of Ω.

3. Generalization

In fact we can extend the result above to a more general case where $f(x)$ is allowed to have discontinuities.

Definition 3.1. We say f has invariant continuity filter in Ω if at every $\omega \in \Omega$, there is a filter F_ω, such that any $A_\omega \in F_\omega$ satisfies the following conditions:

1. $\mu(A_\omega \cap B(\omega,\delta)) > 0$, for any $\delta > 0$,
2. $\lim_{\omega_k \in A_\omega, \omega_k \to \omega} f(\omega_k) \to f(\omega)$,
3. $T^n(A_\omega) \in F_{T^n\omega}$, for any $n \in \mathbb{Z}^\gamma$.

Example. Let $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and μ be the Lebesgue measure for any $\theta \in \Omega$, $T(\theta) = \theta + \alpha$ where $\alpha \notin \mathbb{Q}$ and $f(x) = \{x\}$. Define $(H_\theta u)(n) = u(n+1) + u(n-1) + f(T^n\theta)u(n)$. The reason why we are interested in this model is that ULE has recently been shown for it in [4]. Obviously in this model f is not continuous but it does have continuity invariant filter at every $\theta \in [0,1]$. In fact, the filter at θ is the set of all intervals with left endpoint θ. Generally speaking, all the right or left continuous function defined on $\mathbb{R}$ with direction preserving T has invariant continuity filter at every point.

Now we have the following theorem:

Theorem 3.2. *Assume f is bounded and has invariant continuity filter at every $\omega \in \Omega$. Then if H_ω has ULE in a positive μ-measure set, H_ω has homogenous ULE in $supp(\mu)$.*

As before, we also have:

Theorem 3.2′. *Assume f is bounded and has invariant continuity filter at every $\omega \in \Omega$. Then if H_ω has ULE at a single ω, H_ω has homogenous ULE in Ω.*

Proof. Notice that in the proof of Theorem 1.2, we only use the continuity of f in the last step, which means we still have

$$(\delta_l, e^{-itH_\omega}\delta_m) \leq Ce^{-\alpha|l-m|}$$

for $\omega \in \Omega_0$, where $\mu(\Omega_0) = 1$.

Now consider any $\omega_0 \notin \Omega_0$. We know $\mu(A_{\omega_0} \cap B(\omega_0, \frac{1}{k})) > 0$, hence we can choose $\omega_k^{(0)} \in A_{\omega_0} \cap B(\omega_0, \frac{1}{k})$, $\omega_k^{(0)} \in \Omega_0$ and $|f(\omega_k^{(0)}) - f(\omega_0)| < \frac{1}{2^k}$. Then $T^m \omega_k^{(0)} \in T^m A_{\omega_0}$, for any $m \in \mathbb{Z}^\gamma$. $T^m \omega_k^{(0)} \to T^m \omega_0$, therefore $f(T^m \omega_k^{(0)}) \to f(T^m \omega_0)$. Hence we can choose a subsequence of $\{\omega_k^{(0)}\}$, say $\{\omega_k^{(m)}\}$, such that

$$|f(T^m \omega_k^{(m)}) - f(T^m \omega_0)| < \frac{1}{2^k}.$$

Notice that by the diagonal argument, we can find a sequence $\{\omega_k\}$, satisfying $\omega_k \in \Omega_0$ and $|f(T^j \omega_k) - f(T^j \omega_0)| < \frac{1}{2^k}$ for any $j \in \mathbb{Z}^\gamma$ and $k \geq |j|$.

Now, let us show that for fixed l, m, t:

$$(\delta_l, e^{-itH_{\omega_k}} \delta_m) \to (\delta_l, e^{-itH_{\omega_0}} \delta_m).$$

Indeed: $|(\delta_l, e^{-itH_{\omega_k}} \delta_m) - (\delta_l, e^{-itH_{\omega_0}} \delta_m)|$
$= |(\delta_l, (e^{-it(H_{\omega_k} - H_{\omega_0})} - 1) e^{-itH_{\omega_0}} \delta_m)|$
$= |(\delta_l, (e^{-it(H_{\omega_k} - H_{\omega_0})} - 1) \sum_{r=-\infty}^{\infty} a_r \delta_r)|$
$\leq |a_l| (e^{|t||f(T^l \omega_k) - f(T^l \omega_0)|} - 1) \to 0$, as $k \to \infty$.

Hence $|(\delta_l, e^{-itH_{\omega_0}} \delta_m)| \leq C e^{-\alpha |l-m|}$. □

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