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THE SPECTRA OF SURFACE MARYLAND MODEL FOR ALL PHASES

WENCAI LIU

ABSTRACT. We study the discrete Schrödinger operators $H_{\lambda,\alpha,\theta}$ on $\ell^2(\mathbb{Z}^{d+1})$ with surface potential of the form $V(n, x) = \lambda\delta(x) \tan \pi(\alpha \cdot n + \theta)$, and $H_{\lambda,\alpha,\theta}^+$ on $\ell^2(\mathbb{Z}^d \times \mathbb{Z}_+)$ with the boundary condition $\psi_{(n,-1)} = \lambda \tan \pi(\alpha \cdot n + \theta) \psi_{(n,0)}$, where $\alpha \in \mathbb{R}^d$ is rationally independent. We show that the spectra of $H_{\lambda,\alpha,\theta}$ and $H_{\lambda,\alpha,\theta}^+$ are $(-\infty, \infty)$ for all parameters. We can also determine the absolutely continuous spectra and Hausdorff dimension of the spectral measures if $d = 1$.

1. INTRODUCTION

In this paper, we study the discrete Schrödinger operators with surface potential. Let $d \geq 1$ be given and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. We denote the points in \mathbb{Z}^{d+1} (or $\mathbb{Z}_+^{d+1} = \mathbb{Z}^d \times \mathbb{Z}_+$) by (n, x) , where $n \in \mathbb{Z}^d, x \in \mathbb{Z}$ (or $n \in \mathbb{Z}^d, x \in \mathbb{Z}_+$).

Given $y = (y_1, y_2, \dots, y_k) \in \mathbb{Z}^k$ with $k \geq 1$, denote by $\|y\|_k = \sum_{j=1}^k |y_j|$.

Now we introduce two self-adjoint operators $H_{\lambda,\alpha,\theta}$ and $H_{\lambda,\alpha,\theta}^+$. The operator $H_{\lambda,\alpha,\theta}$ acts on $\ell^2(\mathbb{Z}^{d+1})$ as

$$(1) \quad (H_{\lambda,\alpha,\theta}\psi)_{(n,x)} = \sum_{\|(n-n', x-x')\|_{d+1}=1} \psi_{(n',x')} + \lambda\delta(x) \tan \pi(\alpha \cdot n + \theta) \psi_{(n,x)},$$

where $\lambda \neq 0$ is the coupling, $\alpha \in \mathbb{R}^d$ is the frequency, and θ is the phase. In order to make the potential well defined, we always assume θ satisfies the following condition

$$(2) \quad k \cdot \alpha + \theta \neq \frac{1}{2} \pmod{\mathbb{Z}} \text{ for any } k \in \mathbb{Z}^d.$$

The other operator $H_{\lambda,\alpha,\theta}^+$ on $\ell^2(\mathbb{Z}^d \times \mathbb{Z}_+)$ is given by the following equation,

$$(3) \quad (H_{\lambda,\alpha,\theta}^+\psi)_{(n,x)} = \begin{cases} \sum_{\|(n-n', x-x')\|_{d+1}=1} \psi_{(n',x')}, & \text{if } x > 0; \\ \psi_{(n,1)} + \sum_{\|n'-n\|_d=1} \psi_{(n',0)} + \lambda \tan \pi(\alpha \cdot n + \theta) \psi_{(n,0)}, & \text{if } x = 0. \end{cases}$$

Whenever the meaning is clear within the context, we will drop the dependence on parameters λ, α, θ and d . Thus we write H or H_θ for $H_{\lambda,\alpha,\theta}$, and $\|n' - n\|$ for $\|n' - n\|_d$.

Recall that the Maryland model is given by

$$(h_{\lambda,\alpha,\theta}\psi)_n = \sum_{\|n'-n\|=1} \psi_{n'} + \lambda \tan \pi(\alpha \cdot n + \theta) \psi_n,$$

with $n, n' \in \mathbb{Z}^d$. It has been heavily studied in [4–6, 13]. In particular, for $d = 1$, Jitomirskaya-Liu has obtained the complete description of the three spectral types of Maryland model [10].

Now we want to say more about the surface multidimensional operators. Let H_0 be the discrete Laplacian on $\ell^2(\mathbb{Z}^{d+1})$, and H_0^+ be the discrete Laplacian on $\ell^2(\mathbb{Z}_+^{d+1})$ with Dirichlet boundary condition. More precisely,

$$(H_0\psi)_y = \sum_{\|y'-y\|=1} \psi_{y'},$$

where $y, y' \in \mathbb{Z}^{d+1}$. And

$$(H_0^+ \psi)_{(n,x)} = \sum_{\|(n'-x', n-x)\|=1} \psi_{(n',x')}$$

with $\psi_{(n,-1)} = 0$, where $n \in \mathbb{Z}^d$ and $x \in \mathbb{Z}_+$.

Consider the operator H . The support of the potential is on the hyperplane \mathbb{Z}^d of the space \mathbb{Z}^{d+1} . For the other one H^+ , it is the discrete Laplacian on $\ell^2(\mathbb{Z}_+^{d+1})$ with the boundary condition $\psi_{(n,-1)} = \lambda \tan \pi(\alpha \cdot n + \theta) \psi_{(n,0)}$. Thus it is natural to call both operators H and H_+ the surface Maryland model.

Denote by $\sigma(H)$ and $\sigma_{ac}(H)$ the spectrum, and absolutely continuous spectrum of the self-adjoint operator H . We recall that $\sigma(H_0) = [-2(d+1), 2(d+1)]$ ($\sigma(H_0^+) = [-2(d+1), 2(d+1)]$) and the spectrum of H_0 (H_0^+) is purely absolutely continuous. Denote by $c_d = 2(d+1)$.

We recall that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is rationally independent if for any choice of integers $(k_1, k_2, \dots, k_d) \neq 0$,

$$\sum_{j=1}^d k_j \alpha_j \notin \mathbb{Z}.$$

In the rest of the paper, we always assume $\lambda \neq 0$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is rationally independent and θ satisfies condition (2). We will often use “all $\lambda(\alpha, \theta)$ ”, meaning “all $\lambda(\alpha, \theta)$ as above”.

Now we start to state some known results.

Theorem 1.1. [1, 7] *Let $H_{\lambda,\alpha,\theta}$ and $H_{\lambda,\alpha,\theta}^+$ be given by (1) and (3) respectively. Then for all parameters $(\lambda, \alpha, \theta)$, $[-c_d, c_d] \subset \sigma_{ac}(H_{\lambda,\alpha,\theta}), \sigma_{ac}(H_{\lambda,\alpha,\theta}^+)$, and $H_{\lambda,\alpha,\theta}, H_{\lambda,\alpha,\theta}^+$ has purely absolutely continuous spectrum on the interval $(-c_d, c_d)$.*

We say $\alpha \in \mathbb{R}^d$ satisfies a Diophantine condition if there exists $\kappa > 0$ and $\tau > 0$, such that

$$(4) \quad \|k \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} > \kappa \|k\|^{-\tau} \text{ for any } k \in \mathbb{Z}^d \setminus \{0\},$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_j |x - j|$.

It is to be believed that H and H^+ have purely singular spectra outside the set $[-c_d, c_d]$. Khoruzenko and Pastur [11] have proved this for α satisfying Diophantine condition.

Theorem 1.2. [11] *Assume frequency α satisfies Diophantine condition (4). Then $\sigma(H_{\lambda,\alpha,\theta}) = (-\infty, \infty)$ and $H_{\lambda,\alpha,\theta}$ only has pure point on $(-\infty, \infty) \setminus [-c_d, c_d]$. On this set, the eigenvalues are simple and the corresponding eigenfunctions decay exponentially. These properties also apply to the operator $H_{\lambda,\alpha,\theta}^+$.*

For the surface Maryland model, one of the basic ideas is to integrate the x -variable and to reduce the $d+1$ -dimensional spectral problem to a d -dimensional problem. This technology has been heavily developed by use of Green function in [7–9, 11]. This technology can also be extended to the generalized surface Maryland model [1, 2]. In this paper, we also use this basic technology to analyze the spectra of surface Maryland model. However, we bring a new lighting on the subject.

Our first result is to obtain the explicit formula of the spectra of $H_{\lambda,\alpha,\theta}$ and $H_{\lambda,\alpha,\theta}^+$.

Theorem 1.3. *Suppose α is rationally independent and $\lambda \neq 0$, then $\sigma(H_{\lambda,\alpha,\theta}) = \sigma(H_{\lambda,\alpha,\theta}^+) = (-\infty, +\infty)$ for all θ .*

Remark From Theorems 1.1 and 1.2, we directly know that Theorem 1.3 holds for Diophantine α . Diophantine condition is necessary for Theorem 1.2 when we solve the cohomological equation. Our aim is to prove the Theorem 1.3 for non-Diophantine α .

Secondly we study the absolutely continuous spectrum. By Theorem 1.1, H (H^+) has purely absolutely continuous spectrum in the interval $(-c_d, c_d)$, it was not known if H (H^+) has some absolutely continuous component in the remaining interval $\mathbb{R} \setminus [-c_d, c_d]$. It is believed that H (H^+) has no absolutely continuous component in $\mathbb{R} \setminus [-c_d, c_d]$. In this paper, we solve the problem for $d = 1$. We construct an arithmetically defined measure zero set (actually Hausdorff dimension zero) that supports spectral measures in the regime $\mathbb{R} \setminus [-c_d, c_d]$, which directly implies the following Theorem.

Theorem 1.4. *Let $H_{\lambda, \alpha, \theta}$ and $H_{\lambda, \alpha, \theta}^+$ be given by (1) and (3) respectively with $d = 1$. Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\sigma_{ac}(H_{\lambda, \alpha, \theta}) = \sigma_{ac}(H_{\lambda, \alpha, \theta}^+) = [-4, 4]$ and $H_{\lambda, \alpha, \theta}$, $H_{\lambda, \alpha, \theta}^+$ has purely absolutely continuous spectra on $(-4, 4)$ for all θ . Furthermore, the spectral measures of both operators are of Hausdorff dimension one restricted to $(-4, 4)$ and of Hausdorff dimension zero restricted to $\mathbb{R} \setminus [-4, 4]$ for all θ .*

Remark 1.5. The fact that the spectral measures restricted to $(-4, 4)$ are of Hausdorff dimension one is trivial because $H_{\lambda, \alpha, \theta}$ and $H_{\lambda, \alpha, \theta}^+$ have purely absolutely continuous spectra on $(-4, 4)$.

We will prove Theorems 1.3, 1.4 in two steps. Firstly, we will reduce the $d + 1$ -dimensional spectral problem to a d -dimensional problem in §2. Secondly, we will study the d -dimensional problem using the ideas in paper [10] (§3 and §4), in which the authors deal with the Maryland model.

2. GREEN FUNCTION AND DIMENSION REDUCTION

Because our arguments are almost the same for the operators H and H^+ , we concentrate on H in the following discussion, and only point out the difference for H^+ .

In this section, we carry out the dimension reduction for the surface Maryland model, and lay the ground for our main results. We first consider the operator H . We should point out that all the contents in this section are from [1, 7, 11]. For convenience, we rewrote it here.

Denote by $G(X_1; X_2)$ and $G_0(X_1 - X_2)$ the Green functions of the operators H and H_0 respectively, i.e., the infinite dimensional matrices $(H - zI)^{-1}$ and $(H_0 - zI)^{-1}$ with $\Im z \neq 0$. Then, using the resolvent identity, one has

$$(5) \quad G(X_1; X_2) = G_0(X_1 - X_2) - \sum_{\eta \in \mathbb{Z}^d} G_0(n_1 - \eta, x_1) \lambda v(\eta) G(\eta, 0; X_2),$$

where $X_1 = (n_1, x_1) \in \mathbb{Z}^{d+1}$, $X_2 = (n_2, x_2) \in \mathbb{Z}^{d+1}$ and $v(\eta) = \tan \pi(\alpha \cdot \eta + \theta)$. For simpleness, denote by v (v^{-1}) be the multiplication operator by $v(\eta)$ ($1/v(\eta)$) on $\ell^2(\mathbb{Z}^d)$.

Set $x_1 = 0$ in (5), we find that the function $g_{x_2}(n_1; n_2) = G(n_1, 0; n_2, x_2)$ satisfies

$$(6) \quad g_{x_2}(n_1; n_2) = g_{0, x_2}(n_1 - n_2) - \sum_{\eta \in \mathbb{Z}^d} \Gamma_0(n_1 - \eta) \lambda v(\eta) g_{x_2}(\eta; n_2),$$

where $g_{0, x_2}(\eta) = G_0(\eta, -x_2)$ and $\Gamma_0(\eta) = G_0(\eta, 0)$ with $\eta \in \mathbb{Z}^d$. Next we always regard $\Gamma_0(\eta)$ as a operator on \mathbb{Z}^d .

Fix x_2 , equation (6) becomes a problem in dimension d , and has a symbolic solution $g_{x_2} = (I + \Gamma_0 \lambda v)^{-1} g_{0, x_2}$. Combining with (5), we have

$$(7) \quad \begin{aligned} G(X_1; X_2) &= G(n_1, x_1; n_2, x_2) \\ &= G_0(X_1 - X_2) - \sum_{\eta_1, \eta_2 \in \mathbb{Z}^d} G_0(n_1 - \eta_1, x_1) T(\eta_1, \eta_2) G_0(\eta_2 - n_2, -x_2), \end{aligned}$$

where $T = \lambda v(I + \Gamma_0 \lambda v)^{-1} = (\lambda^{-1} v^{-1} + \Gamma_0)^{-1}$.

Notice that the above derivation of equation (7) is only symbolic, because generally, the operator $(I + \Gamma_0 \lambda v)^{-1}$ is not well defined. However, the following theorem shows (7) is true in fact.

Theorem 2.1. (p.116-p.118, [11]) *Let U denote the multiplication by $e^{2\pi i \alpha \cdot n}$ on $\ell^2(\mathbb{Z}^d)$ and $\delta = e^{2\pi i \theta}$. If $\Im z \cdot \lambda < 0$, then equation (7) holds, where T can be written as*

$$(8) \quad T = \lambda(1 - \delta U)(I - \delta C U)^{-1}(\lambda \Gamma_0 - iI)^{-1},$$

where

$$C = (\lambda \Gamma_0 - iI)^{-1}(\lambda \Gamma_0 + iI).$$

Proof. Suppose we can prove $\lambda \Gamma_0 - iI$ is invertible and $\|C\| < 1$, then equation (8) is well defined, and (7) holds by directly computed. Now we start to prove the facts by using Fourier transformation. Given a vector f_n in $\ell^2(\mathbb{Z}^b)$ with $b = d$ or $b = d + 1$, define a $L^2(\mathbb{T}^b)$ function by the following equation,

$$\hat{f}(y) = \sum_{n \in \mathbb{Z}^b} f_n e^{-2\pi i n \cdot y}.$$

Recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We call $\hat{f}(y)$ be the momentum representation of vector f_n . Under the Fourier transformation, H_0 become a multiplication operator on $L^2(\mathbb{T}^{d+1})$ and denote by \hat{H}_0 . More precisely,

$$(\hat{H}_0 \hat{f})(y) = \left(\sum_{j=1}^{d+1} 2 \cos 2\pi y_j \right) \hat{f}(y).$$

Thus in momentum representation, Green function G_0 has the following simple form,

$$(\hat{G}_0 \hat{f})(y) = \left(\frac{1}{\sum_{j=1}^{d+1} 2 \cos 2\pi y_j - z} \right) \hat{f}(y),$$

with $y \in \mathbb{T}^{d+1}$. And the operator Γ_0 becomes

$$(\hat{\Gamma}_0 \hat{f})(y) = \hat{\Gamma}_0(y) \hat{f}(y),$$

where

$$(9) \quad \hat{\Gamma}_0(y) = \int_{\mathbb{T}} \frac{1}{\sum_{j=1}^{d+1} 2 \cos 2\pi y_j - z} dy_{d+1},$$

with $y \in \mathbb{T}^d$. Notice that $\Im \lambda \hat{\Gamma}_0(y) < 0$, then $\lambda \Gamma_0 - iI$ is invertible (because of $\Im z \cdot \lambda < 0$). Furthermore, C also becomes a multiplication operator in momentum representation,

$$(\hat{C} \hat{f})(y) = \frac{\lambda \hat{\Gamma}_0(y) + i}{\lambda \hat{\Gamma}_0(y) - i} \hat{f}(y).$$

This yields $\|C\| < 1$. □

Now we turn to the operator H^+ . If a operator A generates from H , then denote by A^+ be the corresponding operator generated from H^+ . For example $G^+ = (H^+ - zI)^{-1}$. Using a similar arguments like before, one has

$$(10) \quad G^+(X_1; X_2) = G_0^+(X_1; X_2) - \sum_{\eta \in \mathbb{Z}^d} G_0^+(n_1, x_1; \eta, 0) v(\eta) G^+(\eta, 0; X_2),$$

where $X_1 = (n_1, x_1) \in \mathbb{Z}_+^{d+1}$, $X_2 = (n_2, x_2) \in \mathbb{Z}_+^{d+1}$ and $v(\eta) = \lambda \tan \pi(\alpha \cdot \eta + \theta)$. We should point out that $G_0^+(n_1, x_1; n_2, x_2)$ depends on $n_1 - n_2, x_1, x_2$, this is different from $G_0(n_1, x_1; n_2, x_2)$, which only depends on $n_1 - n_2, x_1 - x_2$.

Set $x_1 = 0$ in (10), we find that the function $g_{x_2}^+(n_1; n_2) = G^+(n_1, 0; n_2, x_2)$ satisfies

$$(11) \quad g_{x_2}^+(n_1; n_2) = g_{0, x_2}^+(n_1; n_2) - \sum_{\eta \in \mathbb{Z}^d} \Gamma_0^+(n_1 - \eta) v(\eta) g_{x_2}^+(\eta; n_2),$$

where $g_{0, x_2}^+(n_1; n_2) = G_0^+(n_1, 0; n_2, x_2)$ and $\Gamma_0^+(\eta) = G_0^+(\eta, 0; 0, 0)$ with $\eta \in \mathbb{Z}^d$.

Fix x_2 , equation (11) becomes a problem in dimension d , and has a symbolic solution $g_{x_2}^+ = (I + \Gamma_0^+ v)^{-1} g_{0, x_2}^+$. Combining with (10), we have

$$(12) \quad \begin{aligned} G^+(X_1; X_2) &= G^+(n_1, x_1; n_2, x_2) \\ &= G_0^+(n_1, x_1; n_2, x_2) - \sum_{\eta_1, \eta_2 \in \mathbb{Z}^d} G_0^+(n_1, x_1; \eta_1, 0) T^+(\eta_1, \eta_2) G_0^+(\eta_2, 0; n_2, x_2), \end{aligned}$$

where $T^+ = v(I + \Gamma_0^+ v)^{-1} = (v^{-1} + \Gamma_0^+)^{-1}$.

We should say more about the operator Γ_0^+ . In momentum representation, Γ_0^+ has the following form [7],

$$(\hat{\Gamma}_0^+ \hat{f})(y) = -r(y, z) \hat{f}(y),$$

where $r(y, z)$ is the root of the quadratic equation

$$X + \frac{1}{X} + \sum_j^d 2 \cos 2\pi y_j = z$$

with $|r(y, z)| < 1$.

By a similar proof above, we have the following theorem.

Theorem 2.2. *Suppose $\Im z \cdot \lambda < 0$, then equation (12) holds and T^+ can be written as*

$$(13) \quad T^+ = \lambda(1 - \delta U)(I - \delta C^+ U)^{-1} (\lambda \Gamma_0^+ - iI)^{-1},$$

where

$$C^+ = (\lambda \Gamma_0^+ - iI)^{-1} (\lambda \Gamma_0^+ + iI).$$

Remark 2.3. Notice that for $z \in \mathbb{C} \setminus [-c_d, c_d]$, the operators Γ_0 and Γ_0^+ are well defined.

3. THE SPECTRA OF SURFACE MARYLAND MODEL

In this section, we will prove that for the surface Maryland model $\sigma(H_{\lambda, \alpha, \theta}) = (-\infty, \infty)$ and $\sigma(H_{\lambda, \alpha, \theta}^+) = (-\infty, \infty)$. From Theorem 1.2, one has that for DC α , $\sigma(H_{\lambda, \alpha, \theta}) = (-\infty, \infty)$, $\sigma(H_{\lambda, \alpha, \theta}^+) = (-\infty, \infty)$ for all $\lambda \neq 0$ and θ . Our main work in this section is to show that the spectrum $\sigma(H_{\lambda, \alpha, \theta})$ does not depend on θ .

Following the discussion of unbounded operators in section VIII [12], it suffices to show the self-adjoint operators $H_{\lambda, \alpha, \theta}$ is continuous in norm resolvent sense with respect to θ . Combining with the specific formula of the Green function of surface Maryland model (Theorem 2.1 and Theorem 2.2), we can obtain the results. In [10], the authors have already used this methods to show that the spectrum of Maryland model does not depend on phase θ .

Definition 3.1. Let B_n , $n = 1, 2, \dots$ and B be self-adjoint operators, then B_n is said to converge to B in norm resolvent sense if $G_z(B_n) \rightarrow G_z(B)$ in norm for all z with $\Im z \neq 0$, where $G_z(A) = (A - zI)^{-1}$.

Theorem 3.2. (Theorem VIII.24, [12]) *Let B_n , $n = 1, 2, \dots$ and B be self-adjoint operators, and B_n converge to B in norm resolvent sense. Then for any $E \in \sigma(B)$, there exists $E_n \in \sigma(B_n)$ such that $E_n \rightarrow E$.*

Theorem 3.3. *The spectrum of surface Maryland model does not depend on phase θ . More precisely, set $\sigma(H_{\lambda,\alpha,\theta})$ or $\sigma(H_{\lambda,\alpha,\theta}^+)$ does not depend on phase θ .*

Proof. For given two phases θ_1 and θ_2 , there exists some sequence $j_k \in \mathbb{Z}^d$ such that

$$\lim_{k \rightarrow \infty} \|\theta_1 + j_k \cdot \alpha - \theta_2\|_{\mathbb{R}/\mathbb{Z}} \rightarrow 0,$$

since α is rationally independent.

Let $B_k = H_{\lambda,\alpha,\theta_1+j_k \cdot \alpha}$ ($B_k = H_{\lambda,\alpha,\theta_1+j_k \cdot \alpha}^+$) and $B = H_{\lambda,\alpha,\theta_2}$ ($B = H_{\lambda,\alpha,\theta_2}^+$). We must have B_k converge to B in norm resolvent sense. Indeed, if $\lambda \cdot \Im z < 0$, this is easy to see by (7) and (8) ((12) and (13)); if $\lambda \cdot \Im z > 0$, this is also true by the fact $H_{\lambda,\alpha,\theta} = H_{-\lambda,-\alpha,-\theta}$ ($H_{\lambda,\alpha,\theta}^+ = H_{-\lambda,-\alpha,-\theta}^+$).

Applying Theorem 3.2, for any $E \in \sigma(H_{\lambda,\alpha,\theta_2})$ ($\sigma(H_{\lambda,\alpha,\theta_2}^+)$), there exists $E_k \in \sigma(B_k)$ such that $E_k \rightarrow E$. Clearly, the spectrum of operator B_k does not depend on k , i.e., $\sigma(B_k) = \sigma(H_{\lambda,\alpha,\theta_1})$ ($\sigma(B_k) = \sigma(H_{\lambda,\alpha,\theta_1}^+)$) because the shift operator is a unitary operator. Thus we must have

$$\sigma(H_{\lambda,\alpha,\theta_2}) \subseteq \sigma(H_{\lambda,\alpha,\theta_1}),$$

and

$$\sigma(H_{\lambda,\alpha,\theta_2}^+) \subseteq \sigma(H_{\lambda,\alpha,\theta_1}^+).$$

This implies the theorem since θ_1 and θ_2 are arbitrary. \square

The proof of Theorem 1.3

Proof. To avoid repetition, we only need to prove the theorem for operator H . Denote by $\mathbb{E}\{\cdot\}$ the average value over \mathbb{T} . Since $\|C\| < 1$, one has

$$(I - \delta CU)^{-1} = I + \sum_{k=1}^{\infty} (\delta CU)^k.$$

Notice that $\delta U = U\delta$ and $\delta C = C\delta$, we have

$$(I - \delta CU)^{-1} = I + \sum_{k=1}^{\infty} \delta^k (CU)^k.$$

Putting (8) into (7) and integrating over \mathbb{T} with respect to variable θ , one has that $\mathbb{E}\{G\}$ does not depend on α .

Suppose Theorem 1.3 does not hold, then by Theorem 3.3 there exist some $\alpha_1 \in \mathbb{R}^d$ and an interval I such that $I \subset (-\infty, \infty) \setminus \sigma(H_{\lambda,\alpha_1,\theta})$ for all θ . Thus for any given vectors δ_y , $y \in \mathbb{Z}^{d+1}$, the spectral measure $\mu_{\theta,\alpha_1,y}$ determined by

$$(14) \quad ((H_{\lambda,\alpha_1,\theta} - zI)^{-1} \delta_y, \delta_y) = \int \frac{d\mu_{\theta,\alpha_1,y}(\phi)}{\phi - z}$$

satisfies

$$\mu_{\theta,\alpha_1,y}(I) = 0 \text{ for all } \theta.$$

Using the fact that $\mathbb{E}\{G\}$ does not depend on α and (14), one has

$$\int_{\mathbb{T}} \mu_{\theta,\alpha,y}(I) d\theta = \int_{\mathbb{T}} \mu_{\theta,\alpha_1,y}(I) d\theta = 0$$

for any $\alpha \in \mathbb{R}^d$ and $y \in \mathbb{Z}^{d+1}$. Combining with the fact that $\sigma(H_{\lambda,\alpha,\theta})$ does not depend on θ , we have

$$I \subset (-\infty, \infty) \setminus \sigma(H_{\lambda,\alpha,\theta})$$

for all $\alpha \in \mathbb{R}^d$. This is contradicted to Theorems 1.1 and 1.2. \square

4. THE SPECTRAL MEASURES FOR $d = 1$

We always assume $E \in (-\infty, \infty) \setminus [-c_d, c_d]$ below.

Let

$$\mathcal{P}_b = \{ \{ \psi_y \}_{y \in \mathbb{Z}^b} : |\psi_y| \leq C(1 + \|y\|^C) \text{ for some constant } C \},$$

where $b = d$ or $b = d + 1$. Usually, we call elements in \mathcal{P}_b polynomially bounded vectors.

Theorem 4.1. *If the equation*

$$(15) \quad v(n)^{-1} \varphi_n + \sum_{\eta \in \mathbb{Z}^d} \lambda \Gamma_0(n - \eta) \varphi_\eta = 0$$

with $n \in \mathbb{Z}^d$ has a nonzero polynomially bounded solution φ , then

$$(16) \quad \psi_{(n,x)} = \sum_{\eta \in \mathbb{Z}^d} G_0(n - \eta, x) \varphi_\eta$$

solves equation (1) and $\psi_{(n,x)}$ is nonzero and polynomially bounded. If equation (1) has a nonzero polynomially bounded solution, define

$$(17) \quad \varphi_n = \sum_{\eta \in \mathbb{Z}^d} \Gamma_0^{-1}(n - \eta) \psi_{(\eta,0)}.$$

Then φ_n solves equation (15) and φ_n is nonzero and polynomially bounded.

Notice that for $E \in (-\infty, \infty) \setminus [-c_d, c_d]$, the operator Γ_0 is well defined and

$$|\Gamma_0^{-1}(n)|, |\Gamma_0(n)| \leq e^{-t\|n\|}$$

for some $t > 0$ (t depends on the distance between E and $[-c_d, c_d]$). Similarly, for some $t > 0$,

$$|G_0(n, x)| \leq e^{-t(\|n\| + |x|)}$$

with $n \in \mathbb{Z}^d$.

The proof of Theorem 4.1 can be found in [11]. For convenience, we give a short proof in the Appendix.

Now we introduce the Cayley transformation for a self-adjoint operator.

Definition 4.2. Given a self-adjoint operator A on Hilbert space \mathbb{H} , we call operator $(I - iA)(I + iA)^{-1}$ the Cayley transformation for operator A .

Clearly, $(I + iA)^{-1}$ is well defined and $(I - iA)(I + iA)^{-1}$ is a unitary operator.

Lemma 4.3. ([4, 13]) *If a vector $u \in \mathcal{P}_d$ satisfies equation (15), and define $c = (1 + i\lambda\Gamma_0)\varphi$, then $c \in \mathcal{P}_d$ and*

$$(18) \quad (1 + iv^{-1})(1 - iv^{-1})^{-1}c = (1 - i\lambda\Gamma_0)(1 + i\lambda\Gamma_0)^{-1}c.$$

And the converse is true. More concretely, if $c \in \mathcal{P}_d$ and satisfies (18), then the vector $\varphi = (1 + i\lambda\Gamma_0)^{-1}c \in \mathcal{P}_d$ and satisfies equation (15).

It is easy to see that operator $(1 + iv^{-1})(1 - iv^{-1})^{-1}$ is multiplication by

$$-e^{-2\pi i(n \cdot \alpha + \theta)}$$

We extend the Fourier Transform to a sequence $f_n \in \mathcal{P}_d$ in distributional sense. More concretely, for given sequence $f_n \in \mathcal{P}_d$, define the Fourier Transform by the following equation,

$$\hat{f}(y) = \sum_{n \in \mathbb{Z}^d} e^{-2\pi i n \cdot y} f_n.$$

By Fourier transformation, the equation (18) becomes

$$(19) \quad q(y)\hat{c}(y) = e^{-2\pi i\theta}\hat{c}(y + \alpha),$$

where $q(y) = -\frac{1-i\lambda\hat{\Gamma}_0(y)}{1+i\lambda\hat{\Gamma}_0(y)}$. It is easy to see that $q(y)$ is analytic on \mathbb{T}^d , and $|q(y)| = 1$ for $y \in \mathbb{R}^d$. We can rewrite

$$(20) \quad q(y) = e^{-2\pi i\zeta(y)},$$

where ζ is analytic on \mathbb{T}^d . Notice that ζ only depends on λ and E .

Next we will study the ergodicity of the analytical function, the proof is similar to the proof in [10].

Let $\frac{p_n}{q_n}$ be the continued fraction approximants to $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$(21) \quad \forall 1 \leq k < q_{n+1}, \text{dist}(k\alpha, \mathbb{Z}) \geq |q_n\alpha - p_n|,$$

and

$$(22) \quad \frac{1}{2q_{n+1}} \leq \Delta_n := |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}}.$$

We define an index $\beta(\alpha)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$(23) \quad \beta(\alpha) = \limsup_{n \rightarrow \infty} \frac{\ln q_{n+1}}{q_n}.$$

Lemma 4.4. *Let $f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ be a real analytical function on the strip $\{z : |\Im z| \leq \rho\}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Suppose $\beta(\alpha) = 0$. Then for any integer sequence $\{j_k\}$ such that*

$$\lim_{k \rightarrow \infty} \|j_k\alpha\|_{\mathbb{R}/\mathbb{Z}} = 0,$$

we have for $x \in \{z : |\Im z| \leq \frac{\rho}{2}\}$ uniformly,

$$(24) \quad \lim_{k \rightarrow \infty} \left(\sum_{j=0}^{j_k-1} f(x + j\alpha) - j_k \int_{\mathbb{T}} f(x) dx \right) = 0.$$

Proof. By the assumption, one has $|f_k| \leq Ce^{-2\pi\rho|k|}$, where f_k is the Fourier coefficients of $f(x)$.

$$\sum_{m=0}^{j_k-1} f(x + m\alpha) - j_k \int_{\mathbb{T}} f(x) = \sum_{j \neq 0} \frac{1 - e^{-2\pi i j j_k \alpha}}{1 - e^{-2\pi i j \alpha}} f_j e^{-2\pi j x}.$$

By the fact $\|j_k\alpha\|_{\mathbb{R}/\mathbb{Z}} \rightarrow 0$, it suffices to show

$$(25) \quad \sum_{j \neq 0} \sup_k \left| \frac{1 - e^{-2\pi i j j_k \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| e^{\rho\pi|j|} < \infty.$$

By (21),(22) and $\beta(\alpha) = 0$, one has

$$\|j\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq ce^{-\frac{\rho}{4}|j|}.$$

Thus

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j j_k \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq Ce^{\frac{\rho}{4}|j|} e^{-2\pi\rho|j|} \\ &\leq Ce^{-\frac{3\pi\rho}{2}|j|} \end{aligned}$$

This implies (25). □

Now we concern the case $\beta(\alpha) > 0$, by the definition, there exists a sequence $\{q_{n_k}\}$ such that

$$(26) \quad q_{n_k+1} \geq e^{\frac{3}{4}\beta q_{n_k}}.$$

Lemma 4.5. *Let $f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ be a real analytical function on the strip $\{z : |\Im z| \leq \rho\}$. Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, suppose $\beta(\alpha) > 0$ and let $\bar{\rho} = \frac{1}{30} \inf\{\rho, \beta\}$. Let*

$$\{t_m\}_{m=1}^\infty = \{q_{n_k}, 2q_{n_k}, 3q_{n_k}, \dots, \ell_k q_{n_k}\}_{k=1}^\infty,$$

where q_{n_k} given by (26) and $\ell_k = \lfloor e^{\bar{\rho} q_{n_k}} \rfloor^1$. Then for $x \in \{z : |\Im z| \leq \frac{\bar{\rho}}{2}\}$ uniformly,

$$(27) \quad \lim_{m \rightarrow \infty} \left(\sum_{j=0}^{t_m-1} f(x+j\alpha) - t_m \int_{\mathbb{T}} f(x) dx \right) = 0.$$

Proof. First, we have

$$\sum_{j=0}^{t_m-1} f(x+j\alpha) - t_m \int_{\mathbb{T}} f(x) = \sum_{j \neq 0} \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j e^{-2\pi j x}.$$

By (22) and (26), one has

$$\begin{aligned} \|t_m \alpha\|_{\mathbb{R}/\mathbb{Z}} &\leq \lfloor e^{\bar{\rho} q_{n_k}} \rfloor \|q_{n_k} \alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\leq C e^{\bar{\rho} q_{n_k}} e^{-3\bar{\rho} q_{n_k}} \\ &\leq C e^{-2\bar{\rho} q_{n_k}}. \end{aligned}$$

This means $\|t_m \alpha\|_{\mathbb{R}/\mathbb{Z}} \rightarrow 0$, then it suffices to show

$$(28) \quad \sum_{j \neq 0} \sup_m \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| e^{\pi \bar{\rho} |j|} < \infty.$$

Fix some $t_m = \ell q_{n_k}$ with $0 < \ell \leq e^{\bar{\rho} q_{n_k}}$.

Case 1: $|j| = q_n$ for some n .

If $n \leq n_k - 1$, by (22) and (26), one has

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq C |j| \ell \frac{\|q_{n_k} \alpha\|_{\mathbb{R}/\mathbb{Z}}}{\|q_n \alpha\|_{\mathbb{R}/\mathbb{Z}}} |f_j| \\ &\leq C |j| \ell \frac{q_{n_k+1}}{q_{n_k+1}} |f_j| \\ &\leq C |j| e^{\bar{\rho} q_{n_k}} \frac{q_{n_k}}{q_{n_k+1}} |f_j| \leq e^{-2\pi \bar{\rho} |j|}. \end{aligned}$$

If $n = n_k$, by (22) and (26), one has

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq C |j| \ell \frac{\|q_{n_k} \alpha\|_{\mathbb{R}/\mathbb{Z}}}{\|q_{n_k} \alpha\|_{\mathbb{R}/\mathbb{Z}}} |f_j| \\ &\leq C |j| \ell |f_j| \\ &\leq e^{-2\pi \bar{\rho} |j|}. \end{aligned}$$

If $n \geq n_k + 1$, by (22) again, one has

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq C \frac{t_m \|j \alpha\|_{\mathbb{R}/\mathbb{Z}}}{\|j \alpha\|_{\mathbb{R}/\mathbb{Z}}} |f_j| \\ &\leq e^{-2\pi \bar{\rho} |j|}. \end{aligned}$$

¹ $\lfloor \ell \rfloor$ denotes the smallest integer not exceeding ℓ

Case 2: $q_n < |j| < q_{n+1}$ for some n .

If $|j| \geq \frac{1}{\rho} \ln q_{n+1}$, by (21) and (22), we have

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq C q_{n+1} |f_j| \\ &\leq C e^{-2\pi \bar{\rho} |j|}. \end{aligned}$$

If $q_n < |j| < q_{n+1}$ and $|j| \leq \frac{1}{\rho} \ln q_{n+1}$, let $|j| = \ell' q_n + j_0$ with $|\ell'| \leq \frac{\ln q_{n+1}}{q_n \bar{\rho}}$ and $0 \leq |j_0| < q_n$. Assume $j_0 \neq 0$, then by (21) and (22), one has

$$\begin{aligned} \|j\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq \Delta_{n-1} - \ell' \Delta_n \\ &\geq \frac{1}{C q_n}. \end{aligned}$$

Thus we have

$$\begin{aligned} \left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| &\leq C q_n |f_j| \\ &\leq e^{-2\pi \bar{\rho} |j|}. \end{aligned}$$

Assume $j_0 = 0$, then $|j| = \ell' q_n$ with $|\ell'| \leq \frac{\ln q_{n+1}}{q_n \bar{\rho}}$. Applying the same proof as in the Case 1, we also have

$$\left| \frac{1 - e^{-2\pi i j t_m \alpha}}{1 - e^{-2\pi i j \alpha}} f_j \right| < e^{-2\pi \bar{\rho} |j|}.$$

The estimate (28) is easy to obtain from the above cases. \square

The proof of Theorem 1.4

Proof. By Theorem 1.1 and Remark 1.5, it suffices to show that in regime $\mathbb{R} \setminus [-c_d, c_d]$ ($d = 1$), the spectral measures of surface Maryland model are supported on a Hausdorff dimension zero set (we only prove the case for operator H). Let

(29)

$$B = \{E : E \in \mathbb{R} \setminus [-4, 4] \text{ such that equation } H\psi = E\psi \text{ has a polynomial bounded solution}\}.$$

By Sch'nol Theorem [3], B is a support of spectral measures restriction on $(-\infty, \infty) \setminus [-4, 4]$. We will prove that B is a Hausdorff dimension zero set, which implies the Theorem directly. Below is the details.

Let $E \in B$, then there exists a polynomial bounded solution of $\psi_{n,x}$ of equation $H\psi = E\psi$. By Theorem 4.1 and Lemma 4.3, we have (using (19))

$$(30) \quad e^{-2\pi i \zeta(y)} \hat{c}(y) = e^{-2\pi i \theta} \hat{c}(y + \alpha),$$

where $\hat{c}(y)$ is the Fourier transformation of a polynomially bounded vector $c \in \ell^2(\mathbb{Z})$.

Case 1: $\beta(\alpha) = 0$.

For any sequence $\{j_k\}$ such that $\lim_{k \rightarrow \infty} \|j_k \alpha\|_{\mathbb{R}/\mathbb{Z}} = 0$, iterate (30) j_k times, we have

$$e^{-2\pi i \sum_{j=0}^{j_k-1} \zeta(y+j\alpha)} \hat{c}(y) = e^{-2\pi i j_k \theta} \hat{c}(y + j_k \alpha).$$

Notice that $\hat{c}(y + j_k \alpha) \rightarrow \hat{c}(y)$ and combine with (24), one has

$$\lim_{n \rightarrow \infty} \|j_k(\zeta_0(E) - \theta)\|_{\mathbb{R}/\mathbb{Z}} = 0,$$

where $\zeta_0 = \int_{\mathbb{R}/\mathbb{Z}} \zeta(y) dy$. By the arbitrary of j_k , we must have

$$\zeta_0(E) \in \theta + \alpha \mathbb{Z} + \mathbb{Z}.$$

Notice that $\zeta_0(E)$ (non-constance) is analytical for $E \in (-\infty, \infty) \setminus [-4, 4]$, then set B only contains countable set. Which means the Hausdorff dimension of B is zero.

Case 2: $\beta(\alpha) > 0$.

By the definition of $\zeta(y)$, $\zeta(y)$ is analytical on the strip $\{z : |\Im z| \leq \rho(E)\}$, where $\rho(E) > 0$ is uniform with respect to $|E| - 4$. Let $\{t_m\}$ be given by Lemma 4.5 and iterate (30) t_m times, we have

$$e^{-2\pi i \sum_{j=0}^{t_m-1} \zeta(y+j\alpha)} \hat{c}(y) = e^{-2\pi i t_m \theta} \hat{c}(y + t_m \alpha).$$

By (27) and the fact $\|t_m \alpha\|_{\mathbb{R}/\mathbb{Z}} \rightarrow 0$, we get

$$\lim_{m \rightarrow \infty} \|t_m(\zeta_0(E) - \theta)\|_{\mathbb{R}/\mathbb{Z}} = 0.$$

This implies for large k ,

$$\|q_{n_k}(\zeta_0(E) - \theta)\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\frac{\bar{\rho}}{2} q_{n_k}},$$

where $\bar{\rho}$ has a uniformly nonzero lower bound depending on $|E| - 4$ and $\beta(\alpha)$.

It is easy to check the set $\{x \in \mathbb{R} : \|q_{n_k} x\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\frac{\bar{\rho}}{2} q_{n_k}} \text{ for large } k\}$ is a Hausdorff measure zero set. In addition $\zeta_0(E)$ is analytical for $E \in (-\infty, \infty) \setminus [-4, 4]$, we know B is also a Hausdorff measure zero set.

For operator H^+ , we only need to replace the operator G_0, Γ_0 etc. with G_0^+, Γ_0^+ etc. To avoid repetition, we omit the proof. \square

APPENDIX A. THE PROOF OF THEOREM 4.1

The proof of Theorem 4.1

Proof. If φ_n is a nonzero polynomially bounded and solves equation (15), it is easy to compute directly that $\psi_{n,x}$ given by equation (16) is nonzero polynomially bounded and solves equation (1).

To the converse, suppose $\psi_{n,x}$ is nonzero polynomially bounded and solves equation (1), this implies

$$(31) \quad ((H_0 - EI)\psi)_{(n,x)} = -\delta(x)\lambda v(n)\psi_{n,x}.$$

Act $G_0 = (H_0 - EI)^{-1}$ on equation (31), one has

$$(32) \quad \psi_{(n,x)} = - \sum_{\eta \in \mathbb{Z}^d} G_0(n - \eta, x) \lambda v(\eta) \psi_{\eta,0}.$$

Thus $\psi_{\eta,0}$ is a non-zero vector, and let $x = 0$ in (32), we obtain

$$\psi_{(n,0)} = - \sum_{\eta \in \mathbb{Z}^d} \Gamma_0(n - \eta) \lambda v(\eta) \psi_{\eta,0}.$$

This yields that φ_n given by (17) solves equation (15) and is nonzero polynomially bounded. We complete the proof of Theorem 4.1. \square

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