WEAK TRIANGLE INEQUALITIES FOR WEAK L_1 NORM

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The weak L_p norm of a random variable X is defined as

$$\|X\|_{p,\infty} = \big(\sup_{t>0} t^p \mathbb{P}\{|X| > t\}\big)^{1/p}, \qquad 0 < p < \infty.$$

It is not a norm but equivalent to a norm if $p > 1$. For $p = 1$ there is no equivalent norm, see a simple example in [1] where

$$\Big\| \sum_{i=1}^N X_i \Big\|_{1,\infty} \sim \log N \quad \text{while all} \quad \|X_i\|_{1,\infty} = 1/N. \tag{1}$$

In this note we discuss two surrogates of triangle inequality for the weak L_1 norm.

Proposition 1. *For any finite sequence of random variables X_i and for any $1 < p < \infty$, one has*

$$\Big\| \Big(\sum_{i=1}^N |X_i|^p \Big)^{1/p} \Big\|_{1,\infty} \le \frac{2p}{p-1} \sum_{i=1}^N \|X_i\|_{1,\infty}.$$

This inequality was proved in [1] for $p = 2$ but the argument there is valid for all $p > 1$. We repeat it below.

Proof. For $M > 0$, consider the truncation $X_{i,M} := X_i \, \mathbf{1}_{\{|X_i| > M\}}$. Then for $u > 0$ we can estimate the tail as

$$\mathbb{P}\Big\{ \Big(\sum_{i=1}^N |X_i|^p \Big)^{1/p} > u \Big\} \le \mathbb{P}\Big\{ \Big(\sum_{i=1}^N |X_{i,M}|^p \Big)^{1/p} > u \Big\} + \sum_{i=1}^N \mathbb{P}\{|X_i| > M\} =: p_1 + p_2.$$

By Chebyshev's ineuquality,

$$p_1 \le \frac{1}{u^p} \, \mathbb{E} \sum_{i=1}^N |X_{i,M}|^p = \frac{1}{u^p} \sum_{i=1}^N \mathbb{E} \, |X_{i,M}|^p.$$

The p-th moment can be estimated using the identity $\mathbb{E}\, Z = \int_0^\infty \mathbb{P}\{Z > t\} \, dt$, which is valid for any non-negative random variable Z. Since $|X_{i,M}|^p$ is bounded by M^p, it yields

$$\mathbb{E} \, |X_{i,M}|^p = \int_0^{M^p} \mathbb{P}\{|X_{i,M}|^p > t\} \, dt \le \int_0^{M^p} \frac{\|X_{i,M}\|_{1,\infty}}{t^{1/p}} \, dt \le \frac{p}{p-1} M^{p-1} \|X_i\|_{1,\infty}.$$

Therefore

$$p_1 \le \frac{1}{u^p} \frac{p}{p-1} M^{p-1} \sum_{i=1}^N \|X_i\|_{1,\infty}.$$

Next, a bound on p_2 follows from Chebychev's inequality:

$$p_2 \le \sum_{i=1}^N \frac{1}{M} \|X_i\|_{1,\infty}$$

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Thus we have proved that

$$p_1 + p_2 \le \Big(\frac{1}{u^p} \frac{p}{p-1} M^{p-1} + \frac{1}{M} \Big) \sum_{i=1}^N \|X_i\|_{1,\infty}.$$

The choice of $M = u$ yields

$$p_1 + p_2 \le \frac{1}{u} \cdot \frac{2p}{p-1} \sum_{i=1}^N \|X_i\|_{1,\infty}.$$

Since this tail bound is valid for all $u > 0$, the proof is complete. □

Let us apply Proposition 1 for p satisfying $(p-1)/p = \log N$ and using that $\sum_{i=1}^N X_i \le e\Big(\sum_{i=1}^N |X_i|^p \Big)^{1/p}$ for this p, we obtain:

Corollary 2. *For any finite sequence of random variables X_i, one has*

$$\Big\| \sum_{i=1}^N X_i \Big\|_{1,\infty} \le 2e \log N \cdot \sum_{i=1}^N \|X_i\|_{1,\infty}.$$

The logarithmic factor is sharp, as can be seen from example (1).

E. Stein and G. Weiss proved a related inequality, see [1].

References

[1] P. Hagelstein, *Weak L^1 norms of random sums*, Proc. Amer. Math. Soc. 133 (2005), 2327–2334