

Basic introductory books on ergodic theory are [31, 48]. More advanced books on ergodic theory are [28, 18].

The Rokhlin Lemma is likely proven in each of the books [31, 18, 28, 48].

The Borel Ergodic Decomposition Theorem is proven independently in [14, 47]. I believe shorter proofs may exist for the weaker result stating that every fixed invariant measure admits an ergodic decomposition, but I don't have a reference for this.

If I remember correctly, a proof of the Becker–Kechris theorem can be found in [17]. Also, the book [17] is a good general reference for dynamics from a descriptive set theoretic point of view.

The theorem that every aperiodic Borel action of a countable group admits a countable generating partition is proven in [22] (relying upon [12] which proved this theorem for actions of \mathbb{Z}). For actions of \mathbb{Z} I believe this was independently proven by Rokhlin and/or Weiss, but I don't remember a reference for this.

Actions of \mathbb{Z} having discrete spectrum are classified in [48], with proofs included.

Properties of Shannon entropy and the Kolmogorov–Sinai entropy theory for actions of \mathbb{Z} are lightly covered in [31, 48]. A nice feature of [48] is it explicitly computes the entropy for several natural examples of \mathbb{Z} actions. Shannon entropy and Kolmogorov–Sinai entropy are covered in more detail in the books [28, 18, 13]. In particular, [18] contains proofs of the Shannon–McMillan–Breiman theorem, Sinai's factor theorem, Ornstein's isomorphism theorem, and Krieger's finite generator theorem. [13] is a great reference for entropy theory in all of its many facets, aside from the entropy theory of actions of non-amenable groups. [28] provides an introduction to the entropy theory of actions of non-amenable groups. Other good references for Sinai's factor theorem and Ornstein's isomorphism theorem are [33] and [11] (this proves the stronger claim about genericity of factor joinings).

Ornstein and other researchers continued to cultivate the techniques used in the isomorphism theorem, ultimately creating 'Ornstein theory,' a collection of practical necessary-and-sufficient conditions for \mathbb{Z} -actions to be isomorphic to Bernoulli. This led to the surprising discovery that many natural examples of \mathbb{Z} -actions turn out to be isomorphic to Bernoulli, such as: factors of Bernoulli shifts [35], inverse limits of Bernoulli shifts [36], ergodic automorphisms of compact metrizable groups [30, 32], mixing Markov chains [15], geodesic flows on surfaces of negative curvature [37], Anosov flows with a smooth measure [40], and two-dimensional billiards with a convex scatterer [19]. These discoveries are generally considered to be the deepest results in entropy theory.

Entropy was extended from actions of \mathbb{Z} to actions of amenable groups by Kieffer in 1975 [29]. The main reference for the entropy theory of actions of amenable groups is [38], which extends Sinai's factor theorem and Ornstein's isomorphism theorem to amenable groups. This paper also included the "Ornstein–Weiss example" of the Bernoulli 2-shift factoring onto the Bernoulli 4-shift for the rank 2 free group.

In [7] Bowen proved that for all non-amenable groups all Bernoulli shifts factor onto one another, improving upon partial results obtained in [2, 3]. The positive direction of Ornstein's isomorphism theorem was extended to all countable groups through the works [46, 6, 45].

For a detailed introduction to sofic groups, see the survey [39]. Sofic entropy was introduced by Bowen [4] for actions of sofic groups admitting a generating partition

having finite Shannon entropy. This restriction was removed, and two equivalent formulations of sofic entropy were presented in [25, 23]. Each of those three papers prove that sofic entropy is an isomorphism invariant. It is shown to extend the classical Kolmogorov–Sinai entropy of actions of amenable groups in [5, 26]. The sofic entropy of Bernoulli shifts was computed in [4, 25, 27]. Kerr proved that Bernoulli shifts over sofic groups have completely positive entropy in [24] and this was generalized to a uniform mixing condition by Tim Austin and Peter Burton [1]. Sofic entropy was computed for algebraic actions and Gaussian actions by Hayes [20, 21].

Rokhlin entropy was introduced in [41] and in [42] Rokhlin entropy is related to the open problems of classifying Bernoulli shifts, Gottschalk’s surjunctivity conjecture, and Kaplansky’s direct finiteness conjecture. Naive entropy and percolation entropy are shown to be upper-bounds to Rokhlin entropy in [43] (for more on naive entropy see [10]). Sinai’s factor theorem for general countable groups is proven in [44]. In [8] Bowen proved that every free action of a non-amenable group is the factor of an action having 0 Rokhlin entropy, and that for all countably infinite groups the generic action has 0 Rokhlin entropy.

Surveys on the entropy theory for non-amenable groups can be found in [28, 9, 16].

REFERENCES

- [1] T. Austin and P. Burton, *Uniform mixing and completely positive sofic entropy*, to appear in Journal d’Analyse Mathématique.
- [2] K. Ball, *Factors of independent and identically distributed processes with non-amenable group actions*, Ergodic Theory and Dynamical Systems 25 (2005), no. 3, 711–730.
- [3] L. Bowen, *Weak isomorphisms between Bernoulli shifts*, Israel J. of Math. 183 (2011), no. 1, 93–102.
- [4] L. Bowen, *Measure conjugacy invariants for actions of countable sofic groups*, Journal of the American Mathematical Society 23 (2010), 217–245.
- [5] L. Bowen, *Sofic entropy and amenable groups*, Ergod. Th. & Dynam. Sys. 32 (2012), no. 2, 427–466.
- [6] L. Bowen, *Every countably infinite group is almost Ornstein*, Dynamical systems and group actions, 67–78, Contemp. Math., 567, Amer. Math. Soc., Providence, RI, 2012.
- [7] L. Bowen, *Finitary random interlacements and the Gaboriau-Lyons problem*, preprint. <https://arxiv.org/abs/1707.09573>.
- [8] L. Bowen, *Zero entropy is generic*, Entropy 18 (2016), no. 6, paper no. 220, 20 pp.
- [9] L. Bowen, *Examples in the entropy theory of countable group actions*, preprint. <https://arxiv.org/abs/1704.06349>.
- [10] P. Burton, *Naive entropy of dynamical systems*, preprint. <http://arxiv.org/abs/1503.06360v2>.
- [11] R. Burton, M. Keane, and J. Serafin, *Residuality of dynamical morphisms*, Colloquium Mathematicae (84 / 85), no. 2, 307 – 317, 2000.
- [12] R. Dougherty, S. Jackson, and A. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. (341), 193–225, 1994.
- [13] T. Downarowicz, *Entropy in Dynamical Systems*. Cambridge University Press, New York, 2011.
- [14] R. H. Farrell, *Representation of invariant measures*, Illinois J. Math. 6 (1962), 447–467.
- [15] N. A. Friedman and D. Ornstein, *On isomorphism of weak Bernoulli transformations*, Advances in Math 5 (1970), 365–394.
- [16] D. Gaboriau, *Entropie sofique, d’apres Lewis Bowen, David Kerr et Hanfeng Li* (French), Bourbaki seminar, Jan. 2016. <http://perso.ens-lyon.fr/gaboriau/Travaux-Publi/Bourbaki-ent-sofique/Gaboriau-Entropie-sofique-Bourbaki.pdf>.
- [17] S. Gao, *Invariant Descriptive Set Theory*. CRC Press, 2008.
- [18] E. Glasner, *Ergodic Theory Via Joinings*. American Mathematical Society, 2003.

- [19] G. Gallavotti and D. Ornstein, *Billiards and Bernoulli schemes*, Comm. Math. Phys. 38 (1974), 83–101.
- [20] B. Hayes, *Fuglede-Kadison determinants and sofic entropy*, Geometric and Functional Analysis 26 (2016), no. 2, 520–606.
- [21] B. Hayes, *Sofic entropy of Gaussian actions*, to appear in Ergodic Theory and Dynamical Systems.
- [22] S. Jackson, A.S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic 2 (2002), No. 1, 1–80.
- [23] D. Kerr, *Sofic measure entropy via finite partitions*, Groups Geom. Dyn. 7 (2013), 617–632.
- [24] D. Kerr, *Bernoulli actions of sofic groups have completely positive entropy*, Israel J. Math. 202 (2014), 461–474.
- [25] D. Kerr and H. Li, *Entropy and the variational principle for actions of sofic groups*, Invent. Math. 186 (2011), 501–558.
- [26] D. Kerr and H. Li, *Soficity, amenability, and dynamical entropy*, American Journal of Mathematics 135 (2013), 721–761.
- [27] D. Kerr and H. Li, *Bernoulli actions and infinite entropy*, Groups Geom. Dyn. 5 (2011), 663–672.
- [28] D. Kerr and H. Li, *Ergodic Theory: Independence and Dichotomies*. Springer-Verlag, New York, 2016.
- [29] J. C. Kieffer, *A generalized Shannon–McMillan Theorem for the action of an amenable group on a probability space*, Annals of Probability 3 (1975), no. 6, 1031–1037.
- [30] D. Lind, *The structure of skew products with ergodic group actions*, Israel Journal of Math 28 (1977), 205–248.
- [31] R. McCutcheon and S. Kalikow, *An Outline of Ergodic Theory*. Cambridge University Press, New York, 2010.
- [32] G. Miles and R. K. Thomas, *The breakdown of automorphisms of compact topological groups*, Advances in Math. Supplementary Studies Vol. 2 (1978), 207–218.
- [33] D. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Advances in Math. 4 (1970), 337–348.
- [34] D. Ornstein, *Two Bernoulli shifts with infinite entropy are isomorphic*, Advances in Math. 5 (1970), 339–348.
- [35] D. Ornstein, *Factors of Bernoulli shifts are Bernoulli shifts*, Advances in Math. 5 (1971), 349–364.
- [36] D.S. Ornstein, *Ergodic Theory, Randomness, and Dynamical Systems*; Yale University Press: New Haven, CT, USA, 1971.
- [37] D. Ornstein and B. Weiss, *Geodesic flows are Bernoullian*, Israel Journal of Math 14 (1973), 184–198.
- [38] D. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, Journal d’Analyse Mathématique 48 (1987), 1–141.
- [39] V. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic 14 (2008), no. 4, 449–480.
- [40] M. Ratner, *Anosov flows with Gibbs measures are also Bernoulli*, Israel Journal of Math 17 (1974), 380–391.
- [41] B. Seward, *Krieger’s finite generator theorem for actions of countable groups I*, preprint. <http://arxiv.org/abs/1405.3604>.
- [42] B. Seward, *Krieger’s finite generator theorem for actions of countable groups II*, preprint. <https://arxiv.org/abs/1501.03367>.
- [43] B. Seward, *Weak containment and Rokhlin entropy*, preprint. <https://arxiv.org/abs/1602.06680>.
- [44] B. Seward, *Positive entropy actions of countable groups factor onto Bernoulli shifts*, in preparation.
- [45] B. Seward, *Bernoulli shifts with bases of equal entropy are isomorphic*, in preparation.
- [46] A. M. Stepin, *Bernoulli shifts on groups*, Dokl. Akad. Nauk SSSR 223 (1975), no. 2, 300–302.
- [47] V. S. Varadarajan, *Groups of automorphisms of Borel spaces*, Trans. Amer. Math. Soc. 109 (1963), 191–220.
- [48] P. Walters, *An Introduction to Ergodic Theory*. Springer-Verlag, New York, 2000.