Algebra Qualifying Exam , June 2009 (10 points each problem)

1 Let D_{2n} be the dihedral group of order $2n$.

(a) Prove that if p is an odd prime, then a Sylow p-subgroup of D_{2n} is normal and cyclic.

(b) Prove that if $2n = 2^{\alpha} \cdot k$ where k is odd then the number of Sylow 2-subgroups of D_{2n} is k. Describe all these subgroups.

2 Let G be a group such that $Aut(G)$ is cyclic. Show that G is abelian.

3 Let $\mathbb{Z}$ be the ring of integers, $\mathbb{F}_5$ be the field with five elements.

(a) Determine whether the rings $\mathbb{F}_5[x]/(x^2+1)$ and $\mathbb{F}_5[x]/(x^2+2)$ are isomorphic.

(b) List all ideals in the ring $\mathbb{Z}[x]/(2, x^3+1)$.

4 Prove that the Galois group of the polynomial $x^5 - 2$ over $\mathbb{Q}$ is isomorphic to the group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in \mathbb{F}_5$ and $a \neq 0$.

5 Let F be a field of characteristic not dividing n. Show that the matrix equation $XY - YX = I_n$ has no solutions, where X and Y are unknown $n \times n$ matrices with entries in F and I_n is the identity matrix.

6 Let T be a linear operator on a finite dimensional vector space V over $\mathbb{Q}$ such that $T^{15} = I$. Assume that both T^3 and T^5 have no non-zero fixed points in V. Show that the dimension of V is divisible by 8.

7 Let A be a finite Abelian group, p be a prime dividing $|A|$ and k be largest such that p^k divides $|A|$. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes A$ is isomorphic to the Sylow p-subgroup of A.

8 Consider complex representations of the finite group S_4 up to isomorphism.

(a) Show that S_4 has exactly two one dimensional complex representations.

(b) Prove that its other pairwise non-isomorphic complex representations have dimensions 2, 3, and 3.

9 Let R be a commutative local ring with maximal ideal M.

(a) Show that if $x \in M$, then $1 - x$ is invertible.

(b) Show that if in addition that R is Noetherian and I is an ideal satisfying $I^2 = I$, then $I = 0$.

10 Let $\mathbb{F}_q$ be a finite field of q elements. Show that every element $x \in \mathbb{F}_q$ can be written as a sum of two squares in $\mathbb{F}_q$, that is, $x = y^2 + z^2$ for some $y, z \in \mathbb{F}_q$.