Algebra Qualifying Exam , September 2009 (10 points each problem)

1 Prove that there are precisely four groups of order 28 up to isomorphism. How many of them are non-abelian?

2 Prove that there are no simple groups of order 30.

3 (a) Give an example of an infinite group in which every element has finite order.

(b) How many solutions does the equation $x^n + \cdots + x + 1 = 0$ have in a finite field $\mathbb{F}_q$?

4 Let A be a commutative ring with identity. Let S be a non-empty multiplicative subset of A such that $0 \notin S$. Let P be an ideal of A, which is maximal in the set of all ideals that do not intersect S. Prove that P is a prime ideal.

5 Let R be a commutative ring with identity. Let A, B be two $n \times n$ square matrices with entries in R. Show that for variable t,

$$\det(I - ABt) = \det(I - BAt).$$

6 Let $p > 2$ be a prime number. Let T be a linear operator on a finite dimensional vector space V over $\mathbb{Q}$ of dimension not divisible by $p - 1$. Show that $T^{p-1} + \cdots + T + I \neq 0$, where I is the identity map on V.

7 Let K_1 and K_2 be two extension fields of a given field K. Assume that K_1 is a finite and separable extension of K. Show that $K_1 \otimes_K K_2$ is a direct sum of fields as K-algebra.

8 Consider complex representations of the finite group G up to isomorphism.

(a) Show that if G is abelian, then every irreducible representation of G has degree 1.

(b) Show that the number of degree 1 representations of G is equal to $G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G.

9 Suppose that F is an algebraically closed field. Find all monic separable polynomials $f(x) \in F[x]$ such that the set of zeros of $f(x)$ in F is closed under multiplication.

10 Compute the Galois group of the polynomial $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$.