Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n : n \in \mathbb{N})$ and f be real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. Suppose $f_n \xrightarrow{\mu} f$ on D, that is, $(f_n : n \in \mathbb{N})$ converges to f in measure μ on D. Let F be a real-valued uniformly continuous function on $\mathbb{R}$. Show that $F \circ f_n \xrightarrow{\mu} F \circ f$ on D.

2.
   - Let $f \in \mathrm{L}^p([0, 10]),\ p \geq 1$. Prove that $\lim_{t \downarrow 1} (t-1)^{\frac{1}{p}-1} \int_1^t f(s)\, \mathrm{d}s = 0$.
   - Suppose $\int_0^\infty x^{-2} |f|^5 \mathrm{d}x < \infty$. Prove that $\lim_{t \downarrow 0} t^{-\frac{6}{5}} \int_0^t f(x) \mathrm{d}x = 0$.

3. Let $f_n(x) = \frac{n}{2} \chi_{[-\frac{1}{n}, \frac{1}{n}]}$. Prove that for $g \in L^1(\mathbb{R})$,

$$\int \left| \int f_n(y-x) g(x) dx - g(y) \right| dy \to 0 \text{ as } n \to \infty.$$

4. Suppose f is a bounded nonnegative function on (X, μ) with $\mu(X) = \infty$. Show that f is integrable if and only if

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \mu\{x \in X : f(x) > \frac{1}{2^n}\} < \infty$$

5. Let f be an element and $(f_n : n \in \mathbb{N})$ be a sequence in $L^p(X, \mathcal{A}, \mu)$ where $p \in [1, \infty)$ such that $\lim_{n \to \infty} \|f_n - f\|_p = 0$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have

$$\int_E |f_n|^p \, d\mu < \epsilon \quad \text{for every } E \in \mathcal{A} \text{ such that } \mu(E) < \delta.$$

6. Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let Σ_0 be a sub-σ-algebra of Σ. Given an integrable function f on (X, Σ, μ), show that there is a Σ_0-measurable function f_0, such that

$$\int_X fg \, \mathrm{d}\mu = \int_X f_0 g \, \mathrm{d}\mu$$

for every Σ_0-measurable function g such that fg is integrable. (Hint: Use the Radon-Nikodym Theorem.)