Name: ____

Algebra Advisory Examination

RH 114, 9am-11:30am, September 16, 2009

There are 10 problems, each worth 10 points. We prefer complete solutions of a few problems to many partial solutions.

Show all details, and quote properly all theorems you use.

If you use any additional pages for your work, be sure to write your name on each one.

Question	Score
1	
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(1) Let V and W be vector spaces of dimensions m and n respectively, and let $T : V \to W$ be a linear transformation between them. Suppose there is a non-zero element f of the dual space W^* of W such that the equation $w = Tv$ has a solution if and only if $f(w) = 0$. Find the dimension of the kernel of T.

(2) Let $\mathbb{Z}$ denote the ring of integers, and consider the three commutative rings

$$R_1 = \mathbb{Z} \times \mathbb{Z},$$
$$R_2 = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\},$$
$$R_3 = \mathbb{Z}[x]/x^2\mathbb{Z}[x].$$

For $1 \leq i < j \leq 3$, either prove that R_i and R_j are isomorphic rings, or prove that they are not.

(3) Let G be a finite abelian group. Its dual (or character) group $\hat{G}$ is defined to be the group $\mathrm{Hom}(G, \mathbb{C}^*)$ of homomorphisms of G into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers. Elements of $\hat{G}$ are called characters of G.

(a) Show that $\hat{G}$ is equal to the group of homomorphisms of G into the complex roots of unity.

(b) If G is a cyclic group, say of order k, show that $\hat{G}$ is cyclic of order k. (Hint: If g is a generator of G then show that a character $\chi \in \hat{G}$ is uniquely determined by $\chi(g)$. What are the possible values of $\chi(g)$?)

(c) If G is the direct product of subgroups G_1 and G_2, show that $\hat{G}$ is naturally the direct product of subgroups that can be identified with $\hat{G}_1$ and $\hat{G}_2$.

(d) Deduce from (b) and (c) that any finite abelian group G is isomorphic to its dual group $\hat{G}$.

(4) (a) Give an example of a square matrix A, with rational coefficients, having characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$.

(b) Suppose that A and B are square matrices with coefficients in a field k, and both A and B have characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$. Show that A and B are similar, i.e., there is an invertible matrix P with coefficients in k such that $PAP^{-1} = B$.

(5) Suppose R is a commutative ring with identity $1_R \neq 0_R$.
  (a) What does it mean to say that R is an *integral domain*?
  (b) Prove that if R is a finite integral domain, then R is a field.

(6) Suppose G is a group of $351 = 3^3 \cdot 13$. Show that G is not simple. That is, show that G has a proper normal subgroup.

(7) Let R be a commutative ring with identity $1_R \neq 0_R$. Show that R contains a minimal prime ideal P, i.e. a prime ideal which contains no smaller prime ideal.

(8) Let H be a finite dimensional vector space over the complex field $\mathbb{C}$ with a positive definite hermitian inner product $\langle x, y\rangle$.

(a) What are the defining properties of the inner product?

(b) If $T : H \to H$ is a linear transformation, what does it mean to say that T is self-adjoint?

(c) What does it mean to say that a complex $n \times n$ matrix (T_{ij}) is hermitian? Show that $T : H \to H$ is self-adjoint if and only if its matrix with respect to an orthonormal basis for H is hermitian.

(d) If $T : H \to H$ is self-adjoint and V is a T-invariant subspace of H then show that its orthogonal complement $V^{\perp}$ is also T-invariant.

(e) If $T : H \to H$ is self-adjoint, show that any eigenvalue λ of T must be real, and if E_λ is the eigenspace of λ then $E_\lambda^{\perp}$ is invariant under T.

(f) Using the above facts, sketch a proof of the Spectral Theorem: namely, show that if $T : H \to H$ is self-adjoint, then H is the orthogonal direct sum of the eigenspaces of T. (Hint: First show that there exists at least one eigenvalue. Then use part (e) and induction.)

(9) Suppose that V is a vector space, and let $\mathrm{GL}(V)$ be the group of all invertible linear transformations from V to itself. Suppose G is a subgroup of $\mathrm{GL}(V)$, and define

$$R = \{\text{all linear transformations } T : V \to V \text{ such that } T(g(v)) = g(T(v)) \text{ for every } g \in G \text{ and } v \in V\}.$$

(a) Show that R is a ring.

(b) Suppose further that if W is any subspace of V such that $g(W) \subset W$ for every $g \in G$, then either $W = 0$ or $W = V$. Prove that if $T \in R$ and T is not the zero transformation, then T is invertible and $T^{-1} \in R$. (Hint: If $T \in R$, what can you say about the kernel and image of T?)

(10) Suppose that G is a group, and H is a subgroup of G of finite index.
  (a) Prove that if H is a normal subgroup of G, and if $a \in G$ has finite order that is relatively prime to the index of H in G, then $a \in H$.
  (b) Give an example to show that the conclusion of part (a) is not necessarily true if H is a non-normal subgroup of G.