Algebra Comprehensive Examination

June 16, 2009

The 10 exam Problems: We prefer complete solutions of a few problems to many partial solutions. The value of each probelm is 10 points.

Show all details and quote proper theorems you use.

1 Solvable Group

A group G is called *solvable* if it has a series whose factor groups are all abelian, that is, if there are subgroups

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_k = G$$

such that G_{j-1} is normal in G_j, and G_j/G_{j-1} is a commutative group, for $j = 1, 2, \cdots, k$.

Prove that a groups of order 12 is solvable.

2 Maximal subgroup

Prove that any group G has an abelian subgroup that is maximal among all abelian subgroups of G.

3 Willson's theorem, field theory

Let $p > 2$ be a prime number. Prove that $1 \cdot 2 \cdots (p-1) = -1 (mod\, p)$.

4 Commutative Rings

Show that every maximal ideal in a commutative ring with the unit element is prime.

5 Irreducibility

Prove that the polynomial

$$1 + x + x^2 + x^3 + \cdots + x^{p-1}$$

is irreducible over $\mathbb{Z}$ for any prime p.

6 Scalar Matrices

Suppose that A, B are elements of $M_2(\mathbb{C})$ such that $A^2 = B^3 = I, ABA = B^{-1}$ with $A \neq I$, $B \neq I$. If $D \in M_2(\mathbb{C})$ commutes with A, B. Show that D is a scalar matrix, i.e., a scalar multiple of I.

7 JNF

Let V be a finite dimensional vector space over $\mathbb{C}$. Suppose that $T : V \to V$ is linear and $p \in \mathbb{Q}[x]$.

1. Define $p(T)$.
2. Show that if λ is an eigenvalue of T, then $p(\lambda)$ is an eigenvalue of $p(T)$.
3. Show that if λ is an eigenvalue of $p(T)$, then there is an eigenvalue λ' of T such that $\lambda = p(\lambda')$.

8 Operators

Let V be a finite dimensional real vector space with a positive definite scalar product $\langle *, * \rangle$ and T a linear operator on V. Assume that $TT^* = T^*T$. Let ξ be an eigenvector of T. Prove that $T^*\xi$ and ξ are linearly dependent.

9 Integral Domain

Prove that if R is an integral domain, then a non-zero polynomial $f(x) \in R[x]$ can have at most $deg(f)$ distinct roots in R.

10 Simple groups

Show that there is no simple group of order $858 = 2 \cdot 3 \cdot 11 \cdot 13$.