Advisory Exam in Algebra September 2006

NAME

Do 10 problems, with AT LEAST 2 PROBLEMS FROM EACH SECTION.
Do each problem on the sheet for that problem. Write your name on each
page. Show all
details and quote properly any theorems that you use. All problems are
worth 10 points.
Section G: Groups

G1. Show that no group of order 56 is simple.
G2. Let $G$ be a finite group and let $H \leq G$ and $K \leq G$ and $HK \leq G$. (The symbol $\leq$ means subgroup.)

a) If $h \in H$ and $k \in K$ show that $o(hk) \mid o(H)o(K)$.

b) Let $N \leq G$ be such that $o(N)$ is relatively prime to $o(H)o(K)$. Prove that $HN = KN$ implies $H = K$. 
G3. Let $G$ be a group and suppose that $G' \leq H \leq G$, where $G'$ is the commutator subgroup of $G$. Prove that $H$ is a NORMAL subgroup of $G$. 
Section R: Rings

R4. Let $C = \{ P_u | u \in U \}$ be a chain (totally ordered family under inclusion) of prime ideals in a commutative ring $R$ with identity. Prove that the intersection $I = \bigcap C$ is a prime ideal.
R5. Let \( R \) be the ring of all real-valued functions on the closed interval \([0, 1]\) of \( \mathbb{R} \), under addition and multiplication of functions. For a given \( a \in [0, 1] \), let \( I_a \) be the set of all functions in \( R \) for which \( f(a) = 0 \). Prove that \( I_a \) is a maximal ideal of \( R \).
R6. Prove that a ring $R$ is an integral domain if and only if the ideal $(0)$ is prime.
Section F: Fields

F7. If $R$ is a commutative ring with identity and $I$ is an ideal of $R$, when is $R/I$ a field? Fully justify your answer.
F8. For $p$ prime show that $p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over $\mathbb{Q}[x]$. *Hint:* consider the polynomial $p(x + 1)$. 
Section L: Linear Algebra

L9. If \( \tau \) is a linear operator on a vector space \( V \) and if \( \tau^3 = \tau^2 \), does it follow that \( \tau^2 = \tau \)? Justify your answer.
L10. If $V$ is a vector space $V$ and

$$V = A \oplus B = C \oplus D$$

with $A \cong C$, does it follow that $B \cong D$? Justify your answer.
L11. Are there complex $n \times n$ matrices $A$ and $B$ for which

$$AB - BA = I$$

Justify your answer.