Advisory Exam in Algebra September 2006

NAME____________________________________

Do 10 problems, with AT LEAST 2 PROBLEMS FROM EACH SECTION. Do each problem on the sheet for that problem. Write your name on each page. Show all details and quote properly any theorems that you use. All problems are worth 10 points.

NAME____________________________________

Section G: Groups

G1. Show that no group of order 56 is simple.

NAME________________________________

G2. Let G be a finite group and let $H \leq G$ and $K \leq G$ and $HK \leq G$. (The symbol $\leq$ means subgroup.)

a) If $h \in H$ and $k \in K$ show that $o(hk) \mid o(H)o(K)$.

b) Let $N \leq G$ be such that $o(N)$ is relatively prime to $o(H)o(K)$. Prove that $HN = KN$ implies $H = K$.

NAME____________________________________

G3. Let G be a group and suppose that $G' \leq H \leq G$, where G' is the commutator subgroup of G. Prove that H is a NORMAL subgroup of G.

NAME________________________________

Section R: Rings

R4. Let $\mathcal{C} = \{P_u \mid u \in U\}$ be a chain (totally ordered family under inclusion) of prime ideals in a commutative ring R with identity. Prove that the intersection $I = \bigcap \mathcal{C}$ is a prime ideal.

NAME_______________________________

R5. Let R be the ring of all real-valued functions on the closed interval $[0, 1]$ of $\mathbb{R}$, under addition and multiplication of functions. For a given $a \in [0, 1]$, let I_a be the set of all functions in R for which $f(a) = 0$. Prove that I_a is a maximal ideal of R.

NAME______________________________

R6. Prove that a ring R is an integral domain if and only if the ideal $\langle 0 \rangle$ is prime.

NAME________________________________

Section F: Fields

F7. If R is a commutative ring with identity and I is an ideal of R, when is R/I a field? Fully justify your answer.

NAME______________________________

F8. For p prime show that $p(x) = 1 + x + x^2 + \cdots + x^{p-1}$ is irreducible over $\mathbb{Q}[x]$. *Hint*: consider the polynomial $p(x + 1)$.

NAME________________________________

Section L: Linear Algebra

L9. If τ is a linear operator on a vector space V and if $\tau^3 = \tau^2$, does it follow that $\tau^2 = \tau$? Justify your answer.

NAME____________________________________

L10. If V is a vector space V and

$$V = A \oplus B = C \oplus D$$

with $A \approx C$, does it follow that $B \approx D$? Justify your answer.

NAME_______________________________

L11. Are there complex $n \times n$ matrices A and B for which

$$AB - BA = I$$

Justify your answer.