TRY ALL 10 PROBLEMS.

G1. Prove that any group of order $2p$, where $p$ is a prime is solvable. A group $G$ is **solvable** if there is a sequence $\{1\} \leq H_1 \leq \cdots \leq H_n = G$ of subgroups of $G$ where each subgroup $H_i$ is normal in the next subgroup $H_{i+1}$ and the quotient $H_{i+1}/H_i$ is abelian.
G2. Prove that $13^{20} - 1$ is divisible by 33.
G3. Let $G$ be a group of order $p^nm$, where $p \nmid m$. Let $S \in \text{Syl}_p(G)$. Prove that if $S$ is self-normalizing, that is, if $S = N_G(S)$, then $n_p = m$. (As always $n_p$ denotes the number of Sylow $p$-subgroups of $G$.pagination
R4. Let $R$ be a commutative ring with identity and with a unique maximal ideal $M$. Show that every nonunit $x \in R$ is in $M$. 
R5. Let $I$ be an ideal of a commutative ring $R$ with identity and let

$\text{Nil}(I) = \{ r \in R \mid r^n \in I \text{ for some } n \geq 0 \}$

a) Prove that $\text{Nil}(I)$ is an ideal of $R$.
b) Prove that $\text{Nil(}\text{Nil}(I)) = \text{Nil}(I)$. 

F7. Let $F$ and $E$ be finite fields with $F < E$. Prove that the cardinality of $E$ is an integral power of the cardinality of $F$, that is, $|E| = |F|^k$ for some integer $k$. 
L8. Let $S$ be a proper subspace of a finite-dimensional vector space $V$ and let $v \in V \setminus S$. Show that there is a linear functional $f \in V^*$ for which $f(v) = 1$ and $f(s) = 0$ for all $s \in S$. 
L9. Let $V$ be a finite-dimensional vector space and let $\tau \in \mathcal{L}(V)$. Suppose that the minimal polynomial of $\tau$ is $p(x) = x^3 - 2x^2 - x + 2$. Prove that $\tau$ is diagonalizable.
L10. Let $S$ be a subspace of a finite-dimensional inner product space $V$. Prove that each coset in $V/S$ contains exactly one vector that is orthogonal to $S$. 