Advisory Exam in Algebra Sept 2008 **NAME**____________________________________

TRY ALL 10 PROBLEMS.

G1. Prove that any group of order $2p$, where p is a prime is solvable. A group G is **solvable** if there is a sequence $\{1\} \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$ of subgroups of G where each subgroup H_i is normal in the next subgroup H_{i+1} and the quotient H_{i+1}/H_i is abelian.

Advisory Exam in Algebra Sept 2008 **NAME**____________________________________

G2. Prove that $13^{20} - 1$ is divisible by 33.

G3. Let G be a group of order $p^n m$, where $p \not| m$. Let $S \in \mathrm{Syl}_p(G)$. Prove that if S is self-normalizing, that is, if $S = N_G(S)$, then $n_p = m$. (As always n_p denotes the number of Sylow p-subgroups of G.)

Advisory Exam in Algebra Sept 2008 **NAME**______________________________

R4. Let R be a commutative ring with identity and with a unique maximal ideal M. Show that every nonunit $x \in R$ is in M.

Advisory Exam in Algebra Sept 2008 **NAME**____________________________________

R5. Let I be an ideal of a commutative ring R with identity and let

$$\mathrm{Nil}(I) = \{r \in R \mid r^n \in I \text{ for some } n \geq 0\}$$

a) Prove that $\mathrm{Nil}(I)$ is an ideal of R.
b) Prove that $\mathrm{Nil}(\mathrm{Nil}(I)) = \mathrm{Nil}(I)$.

F6. $F < E$ and $F < K$ be field extensions. Let $[E : F] = n$ and $[K : F] = m$ where m and n are relatively prime. Prove that $[EK : F] = mn$.

F7. Let F and E be finite fields with $F < E$. Prove that the cardinality of E is an integral power of the cardinality of F, that is, $|E| = |F|^k$ for some integer k.

L8. Let S be a proper subspace of a finite-dimensional vector space V and let $v \in V \setminus S$. Show that there is a linear functional $f \in V^*$ for which $f(v) = 1$ and $f(s) = 0$ for all $s \in S$.

L9. Let V be a finite-dimensional vector space and let $\tau \in \mathcal{L}(V)$. Suppose that the minimal polynomial of τ is $p(x) = x^3 - 2x^2 - x + 2$. Prove that τ is diagonalizable.

L10. Let S be a subspace of an finite-dimensional inner product space V. Prove that each coset in V/S contains *exactly one* vector that is orthogonal to S.