Algebra Compr. Examination

Sept 14 , 2004

The 10 exam Problems: We prefer complete solutions of a few problems to many partial solutions. The value of each part of all problems is specified at the beginning.

Do each problem on a separate page. Write your name on each page. If you don't understand some terminology, please ask. The notation throughout a given problem remains constant. Show all details and quote properly theorems you use. To pass this exam you need to collect 70 points or higher.

1 Cubics

(8 points) Show that all homogeneous cubic (degree 3) forms in 3 variables form a vector space, compute its dimension and show a basis in it.

2 Conjugacy Classes

(10 points) Let G_1 and G_2 be two finite groups with the same number of conjugacy classes and the same number of elements in each conjugacy class. Are G_1 and G_2 isomorphic? Please explain.

3 Groups of order 6

(10 points)

Describe all groups of order 6.

Hint: Use Sylow's theorem.

4 Simple Groups

(12 points) Prove that A_n,($n > 5$) is a simple group.

5 Abelian groups

(10 points) Describe up to isomorphism all abelian groups of order 144.

6 The group A_4

(10 points) Let A_4 be the alternating group on 4 elements.

4 6.a) Show that A_4 is a solvable group.

2 6.b) Find the center of A_4.

4 6.c) Are there any other nonabelian groups of order 12?

7 JNF

(10 points) Let T be a 3×3 matrix with complex coefficients. Describe all possible solutions of the equation $T^3 = T$.

Hint: Use JNF theorem.

8 Group Actions

(12 points) Consider the permutation action of the group S_4 on the 4 standard basis vectors in a complex 4 dimensional vector space. This action defines a homomorphisn $\rho : S_4 \to GL(4, \mathbb{C})$.

2 8.a) Show that ρ has a trivial kernel.

4 8.b) Show that there exists a vector x - a common eigenvector for all images of elements of S_4 under ρ.

6 8.c) Show that the orthogonal complement to x is also fixed under the action given by $\rho : S_4 \to GL(4, \mathbb{C})$.

9 Centers

5 9.a) Find the center of the group $GL(n, C)$ for $n > 1$.

5 9.b) Find the center of the group $SL(n, C)$ for $n > 1$.

10 Matrices

(8 points) Let A and B be any $n \times n$ matrices over $\mathbb{C}$. Is it possible that $ABA - BAB = E$? Please explain. Here E is the identity matrix.