Algebra Comprehensive Exam

September, 2011

NAME	
SIGNATURE	

- This is a closed-book test. You have 2 hours and 30 minutes to complete the exam.
- The test contains 10 problems. Each problem is worth 10 points.
- **Show all details and quote any theorem you use. We prefer complete solutions of a few problems to many partial solutions.**
- *Please, write clearly and legibly.* Clearly indicate scratch work so it won't be graded.

Problem	1	2	3	4	5	6	7	8	9	10
Score										

Problem 1

Let R be a commutative integral domain with unity. A nonzero, non-unit element $s \in R$ is said to be *special* if, for every element $a \in R$, there exist q, $r \in R$ with $a = qs + r$ and such that r is either 0 or a unit of R.

(1) If $s \in R$ is special, prove that the principal ideal (s) generated by s is maximal in R. (5 points)

(2) Show that every polynomial in $\mathbb{Q}[x]$ of degree 1 is special in $\mathbb{Q}[x]$. (5 points)

Problem 2

A linear transformation $T\colon V \to W$ is said to be *independence preserving* if $T(I) \subset W$ is linearly independent whenever $I \subset V$ is a linearly independent set. Show that T is independence preserving if and only if T is one-to-one. (10 points)

Problem 3

The group $GL_2(\mathbb{C})$ acts on the set $\mathbb{C}^{2\times 2}$ of 2×2 matrices by conjugation. Classify the orbits of this action.

Problem 4

Let n be a positive integer and let $\mathbb{F}_q$ denote the finite field of q elements.

(1) Show that $|GL_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1)$. (5 points)

(2) Compute the order of group $SL_n(\mathbb{F}_q)$. (5 points)

Problem 5

Show that there are no simple groups of order pq, where p and q are primes (not necessarily distinct). (10 points)

Problem 6

Let A and B be two $n \times n$ complex matrices. Let I_n be the $n \times n$ identity matrix.

- Assume that A is non-singular. Show that $\det(I_n - AB) = \det(I_n - BA)$. (5 points)

- Show that $\det(I_n - AB) = \det(I_n - BA)$ always holds, even if A and B are singular. (5 points)

Problem 7

Let R be a commutative ring of characteristic $p > 0$, where p is a prime number. Assume that $a \in R$ is nilpotent, i.e., $a^k = 0$ for some positive integer k. Show that $1 + a$ is unipotent, i.e., some power of $1 + a$ is equal to 1. (10 points)

Problem 8

Let f be a polynomial of degree $n > 0$ over a field F. Let K_f be a splitting field for f over F, that is, K_f is obtained by adjoining all the roots of f in an algebraic closure of F. Show that the extension degree $[K_f : F]$ divides $n!$ (hint: use induction on n). (10 points)

Problem 9

Let H and K be two subgroups of a group G. Show that the subset $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$. (10 points)

Problem 10

Prove that the two polynomials $f(x) = x^5 - 5x \pm 1$ are irreducible over $\mathbb{Q}$. (10 points)