

ALGEBRA QUALIFYING EXAM

September 14, 2011

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each question is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let \mathbb{Z} denote the integers. Let \mathbb{Q} denote the rational numbers. Let \mathbb{C} denote the complex numbers. Let S_n denote the symmetric group on n letters.

1. Show that $\sqrt[4]{2}$ is not contained in any field L that is Galois over \mathbb{Q} with $\text{Gal}(L/\mathbb{Q}) = S_n$, for any positive integer n . You may use without proof the fact that the Galois group of the polynomial $x^4 - 2$ over \mathbb{Q} is the dihedral group of order 8.
2. Let p be a prime and F be an algebraically closed field of characteristic p . Let $n = p^a m$, where m is a positive integer not divisible by p . How many n -th roots of unity are there in F ? Prove your answer.
3. A commutative ring R with identity $1 \neq 0$ is called *boolean* if $x^2 = x$ for every $x \in R$. Find all boolean integral domains. Prove that every prime ideal in a boolean ring is maximal.
4. Determine the Galois closure F of the field $\mathbb{Q}(\sqrt{1 + \sqrt{2}})$ over \mathbb{Q} . Determine all elements of the Galois group of the extension F/\mathbb{Q} by describing their actions on the generators of F . Also describe G as an abstract group.
5. Suppose G is a finite group of odd order. Show that if H is a normal subgroup of G of order 5, then H is contained in the center of G .
6. Classify all finite groups whose automorphism group is trivial.
7. Suppose F is a field and $f(x) \in F[x]$ is irreducible. Suppose that E is a splitting field over F for $f(x)$, and that for some $\alpha \in E$, we have $f(\alpha) = f(\alpha + 1) = 0$. Show that the characteristic of F is not zero.
8. Let $V \subset \mathbb{C}[X, Y, Z]$ be the 6-dimensional vector space of homogeneous polynomials of degree 2 over \mathbb{C} . (A polynomial is homogeneous of degree 2 if it is a linear combination of monomials each of which has total degree 2, such as XZ or Y^2 .) View V as a representation of S_3 , with S_3 acting by permuting the variables.
 - (a) Give the character table of S_3 (no proof required).
 - (b) What is the character of the representation of S_3 on V ?
 - (c) Express the character of this representation as a sum of irreducible characters.
9. Suppose A is an 8×8 -matrix with entries in \mathbb{C} , such that:
 - $\dim_{\mathbb{C}}(\ker(A - 2I)) = 2$,
 - $\dim_{\mathbb{C}}(\ker(A - 2I)^2) = 3$,
 - $\dim_{\mathbb{C}}(\ker(A - 3I)) = 2$,
 - $\dim_{\mathbb{C}}(\ker(A - 3I)^2) = 4$,
 - $\dim_{\mathbb{C}}(\ker(A - 3I)^3) = 5$,
 - (a) What is the characteristic polynomial of A ?
 - (b) What is the minimal polynomial of A ?
 - (c) What is the Jordan normal form of A ?
 - (d) What is the rational canonical form of A ?
10. True/False. Answer each question true false. It is not necessary to show your work.
 - (a) If G is a group, and every finitely generated subgroup of G is abelian, then G is abelian.
 - (b) If G is a group, and all proper subgroups of G are normal, then G is abelian.
 - (c) If m and n are relatively prime positive integers, then the tensor product $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.
 - (d) If two 4×4 complex matrices have the same minimal and characteristic polynomials, then they are similar.
 - (e) If R is a commutative ring with identity $1 \neq 0$, and x is in every maximal ideal of R , then $1 + x$ is a unit in R .