ALGEBRA

Advisory Exam (September 15, 2010)

Problem	1	2	3	4	5	6	7	8	9	10	Σ
Points											

Student's name:

Problem G1.

Show that any group or order 185 is commutative.

Problem G2.

Let A, B and C be finitely generated abelian groups such that $A \oplus C \simeq B \oplus C$. Show that also $A \simeq B$.

Problem G3.

Show that for any integer $n \geq 1$ the quotient group $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n.

Problem R4.

Show that the quotient ring $\mathbb{Z}[i]/\langle 3 \rangle$ is a field with 9 elements while $\mathbb{Z}[i]/\langle 2 \rangle$ is not a field.

Problem R5.

Suppose that R is a commutative ring with identity which has a unique maximal ideal M. Show that its complement $R \setminus M$ is precisely the set of units (i.e. invertible elements) in R.

Problem L6.

Find all 4×4 matrices A with real coefficients such that $A^3 = I$, where I is the identity matrix.

Problem L7.

Let V, U, W be finite dimensional vector spaces over $\mathbb{C}$. Consider an injective linear map $\phi : V \to U$, a surjective linear map $\psi : U \to W$. Assume that the composition $\psi \circ \phi$ is zero and that $\dim U = \dim V + \dim W$. Show that $Ker(\psi) = Im(\phi)$ as subspaces of U.

Problem L8.

Let V be a vector space with inner product (Hermitian or Euclidean) and $N : V \to V$ a normal operator (i.e. commuting with its own adjoint). Show that $Ker(N) = Ker(N^*)$.

Problem F9.

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the subfield of elements which are algebraic over $\mathbb{Q}$. Prove that the field extension $\mathbb{Q} \subset \overline{\mathbb{Q}}$ has infinite degree.

Problem F10.

Let F be a field such that the multiplicative group F^* is finitely generated. Show that F is finite. (HINT: eliminate the case of characteristic zero by proving that $\mathbb{Q}^*$ is not finitely generated. If F has finite characteristic p and $x \in F^*$ is of infinite multiplicative order then note x cannot be algebraic over the finite subfield $\mathbb{F}_p \subset F$ and consider irreducibles in $\mathbb{F}_p[x]$.)