**ALGEBRA**

Advisory Exam (September 15, 2010)

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**Student’s name:**
Problem G1.

Show that any group of order 185 is commutative.
Problem G2.

Let $A$, $B$ and $C$ be finitely generated abelian groups such that $A \oplus C \cong B \oplus C$. Show that also $A \cong B$. 
Problem G3.

Show that for any integer \( n \geq 1 \) the quotient group \( \mathbb{Q}/\mathbb{Z} \) has a unique subgroup of order \( n \).
Problem R4.

Show that the quotient ring $\mathbb{Z}[i]/\langle 3 \rangle$ is a field with 9 elements while $\mathbb{Z}[i]/\langle 2 \rangle$ is not a field.
Problem R5.

Suppose that $R$ is a commutative ring with identity which has a unique maximal ideal $M$. Show that its complement $R \setminus M$ is precisely the set of units (i.e. invertible elements) in $R$. 
Find all $4 \times 4$ matrices $A$ with real coefficients such that $A^3 = I$, where $I$ is the identity matrix.
Problem L7.

Let $V, U, W$ be finite dimensional vector spaces over $\mathbb{C}$. Consider an injective linear map $\phi : V \to U$, a surjective linear map $\psi : U \to W$. Assume that the composition $\psi \circ \phi$ is zero and that $\dim U = \dim V + \dim W$. Show that $\ker(\psi) = \text{im}(\phi)$ as subspaces of $U$. 
Problem L8.

Let $V$ be a vector space with inner product (Hermitian or Euclidean) and $N : V \to V$ a normal operator (i.e. commuting with its own adjoint). Show that $\text{Ker}(N) = \text{Ker}(N^*)$. 
Problem F9.
Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the subfield of elements which are algebraic over $\mathbb{Q}$. Prove that the field extension $\mathbb{Q} \subset \overline{\mathbb{Q}}$ has infinite degree.
Problem F10.

Let $F$ be a field such that the multiplicative group $F^*$ is finitely generated. Show that $F$ is finite. (HINT: eliminate the case of characteristic zero by proving that $\mathbb{Q}^*$ is not finitely generated. If $F$ has finite characteristic $p$ and $x \in F^*$ is of infinite multiplicative order then note $x$ cannot be algebraic over the finite subfield $\mathbb{F}_p \subset F$ and consider irreducibles in $\mathbb{F}_p[x]$.)