Comprehensive Exam (June 14, 2010)

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Student’s name:
Problem G1.

Let $G \to H$ be a surjective homomorphism of finite groups. Show that the image of a Sylow $p$-subgroup in $G$ will be a Sylow $p$-subgroup in $H$. 
Problem G2.

Let $S \subset \mathbb{C}$ be the unit circle in the complex plane, i.e., $S := \{ z \in \mathbb{C} \mid |z| = 1 \}$. Let $G$ be an abelian group (written multiplicatively) and note that since $S$ is also an abelian group under complex multiplication, the set $\hat{G}$ of group homomorphisms $\chi : G \rightarrow S$ is a group under pointwise multiplication. It is called the dual group of $G$ and its elements $\chi$ are called characters of $G$. Now assume that $G$ is finite, say of order $n$ and define an "integral" or average value on the complex vector space of $H(G)$ of complex-valued functions $f : G \rightarrow \mathbb{C}$ by $\int f(g) \, dg := \frac{1}{n} \sum_{g \in G} f(g)$ and use it to define an Hermitian inner-product on $H(G)$ by $< f_1, f_2 > := \int f_1(g) \overline{f_2(g)} \, dg$.

a) Show that for any $f \in H(G)$ and $\gamma \in G$, $\int f(\gamma g) \, dg = \int f(g) \, dg$.

b) Show that $\int \chi_0(g) \, dg = 1$ if $\chi_0$ is the identity character; but $\int \chi(g) \, dg = 0$ for any other character $\chi$ of $G$. (Hint: Use a) and $\chi(\gamma g) = \chi(\gamma) \chi(g)$.)

c) Deduce that the set $\hat{G}$ of characters is an orthonormal subset of $H(G)$. (They are in fact an orthonormal basis, but you don’t have to prove that.) (Hint: Show that for any $\chi \in \hat{G}$ its inverse is given by $\chi^{-1}(g) := \overline{\chi(g)}$.)
Problem G3.

Let $A$ be the group of rationals under addition and $M$ the group of non-zero rationals under multiplication. Determine all homomorphisms

$$\phi : A \rightarrow M$$
Problem R4.

Find all maximal ideals in the ring
(a) \( \mathbb{Z} \), (b) \( \mathbb{R}[x] \), (c) \( \mathbb{Z}/n\mathbb{Z} \).
Problem R5.

Let $M$ be a module over the ring $R = \mathbb{Z}/p^n\mathbb{Z}$ and suppose that the number of elements $|M|$ is finite. Show that $|M| = p^k$ for some integer $k \geq 0$. 
Problem L6.

Let $U, V$ and $W$ be subspaces in a vector space. Is it always true that $U \cap (V + W) = U \cap V + U \cap W$? Give a proof or a counterexample.
Problem L7.

Let $V$ be a finite dimensional vector space with an Hermitian inner-product, $A : V \rightarrow V$ a self-adjoint operator (i.e., $A = A^*$) and $v_1, v_2 \in V$ two eigenvectors with distinct eigenvalues. Show that $\langle v_1, v_2 \rangle = 0$. 
Problem L8.

Let $T$ be a linear operator on a vector space $V$ over a field $F$.

a) Show that if $v_1, \ldots, v_k$ are eigenvectors of $T$ belonging to $k$ distinct eigenvalues $c_1, \ldots, c_k$, then the $v_i$ are linearly independent. (Hint: Assume inductively that $v_1, \ldots, v_{j-1}$ are linearly independent but that $v_j$ depends linearly on them and derive a contradiction.)

b) Deduce that if $V$ is $n$-dimensional and if $\chi(\lambda) := \det(T - \lambda I)$, its characteristic polynomial, has $n$ distinct roots in $F$ then $T$ is diagonalizable.
Problem F9.

Let \( E \subset F \) be a field extension of degree 5 and \( a \in F \) an element which is a root of some cubic polynomial \( g(x) \in E[x] \). Show that \( a \in E \).
Problem F10.

What is the number of irreducible degree 4 polynomials in \( \mathbb{F}_p[x] \), where \( p \) is a prime?