ALGEBRA

Comprehensive Exam (June 14, 2010)

Problem	1	2	3	4	5	6	7	8	9	10	Σ
Points											

Student's name:

Problem G1.

Let $G \to H$ be a surjective homomorphism of finite groups. Show that the image of a Sylow p-subgroup in G will be a Sylow p-subgroup in H.

Problem G2.

Let $\mathbb{S} \subset \mathbb{C}$ be the unit circle in the complex plane, i.e., $\mathbb{S} := \{z \in \mathbb{C} \mid |z| = 1\}$. Let G be an abelian group (written multiplicatively) and note that since $\mathbb{S}$ is also an abelian group under complex multiplication, the set $\hat{G}$ of group homomorphisms $\chi : G \to \mathbb{S}$ is a group under pointwise multiplication. It is called the *dual group* of G and its elements χ are called *characters* of G. Now assume that G is finite, say of order n and define an "integral" or average value on the complex vector space of $H(G)$ of complex-valued functions $f : G \to \mathbb{C}$ by $\int f(g)\, dg := \frac{1}{n} \sum_{g \in G} f(g)$ and use it to define an Hermitian inner-product on $H(G)$ by $< f_1, f_2 >:= \int f_1(g) \overline{f_2(g)}\, dg$.

a) Show that for any $f \in H(G)$ and $\gamma \in G$, $\int f(\gamma g)\, dg = \int f(g)\, dg$.
b) Show that $\int \chi_0(g)\, dg = 1$ if χ_0 is the identity character; but $\int \chi(g)\, dg = 0$ for any other character χ of G. (Hint: Use a) and $\chi(\gamma g) = \chi(\gamma)\chi(g)$.)
c) Deduce that the set $\hat{G}$ of characters is an orthonormal subset of $H(G)$. (They are in fact an orthonormal basis, but you don't have to prove that.) (Hint: Show that for any $\chi \in \hat{G}$ its inverse is given by $\chi^{-1}(g) := \overline{\chi(g)}$.)

Problem G3.

Let A be the group of rationals under addition and M the group of non-zero rationals under multiplication. Determine all homomorphisms

$$\phi : A \to M$$

Problem R4.

Find all maximal ideals in the ring

(a) $\mathbb{Z}$,   (b) $\mathbb{R}[x]$,   (c) $\mathbb{Z}/n\mathbb{Z}$.

Problem R5.

Let M be a module over the ring $R = \mathbb{Z}/p^n\mathbb{Z}$ and suppose that the number of elements $|M|$ is finite. Show that $|M| = p^k$ for some integer $k \geq 0$.

Problem L6.

Let U, V and W be subspaces in a vector space. Is it always true that $U \cap (V + W) = U \cap V + U \cap W$? Give a proof or a counterexample.

Problem L7.

Let V be a finite dimensional vector space with an Hermitian inner-product, $A : V \to V$ a self-adjoint operator (i.e., $A = A^*$) and $v_1, v_2 \in V$ two eigen-vectors with distinct eigenvalues. Show that $\langle v_1, v_2 \rangle = 0$.

Problem L8.

Let T be a linear operator on a vector space V over a field F.
a) Show that if $v_1, \ldots, v_k$ are eigenvectors of T belonging to k distinct eigenvalues $c_1, \ldots, c_k$, then the v_i are linearly independent. (Hint: Assume inductively that $v_1. \ldots, v_{j-1}$ are linearly independent but that v_j depends linearly on them and derive a contradiction.)
b) Deduce that if V is n-dimensional and if $\chi(\lambda) := \det(T - \lambda I)$, its characteristic polynomial, has n distinct roots in F then T is diagonalizable.

Problem F9.

Let $E \subset F$ be a field extension of degree 5 and $a \in F$ an element which is a root of some cubic polynomial $g(x) \in E[x]$. Show that $a \in E$.

Problem F10.

What is the number of irreducible degree 4 polynomials in $\mathbb{F}_p[x]$, where p is a prime?