Advisory/Comprehensive Exam in Real Analysis Spring, 2010

Thursday, June 15, 2010, 9:00-11:30am; RH 114

Exam Packet #

Name (please print): ____________________ ____________________
Last First

Student ID: ___ ___ ___ ___ ___ ___ ___ ___

INSTRUCTIONS

(1) Before going on, please print your name and student ID on this Cover Page. Also, please write your student ID and 'Exam Packet' number – but NOT your name – in the places indicated on each 'Problem Page';

(2) Each of the nine problems is given the same credit, so don't get bogged down on a single problem.

(3) Provide complete answers with your justification (proof), written in complete sentences.

(4) If you justify an answer by using – without proof – a 'known theorem', make it clear to the grader which theorem that is. If it is a theorem with a standard name, give its name. If it does not have a name, or if the same name is applied to variations of the same theorem, carefully state the theorem. In any event, make it clear to the grader why you are justified in applying the theorem; that is, indicate why the hypotheses of that theorem are satisfied in the situation at hand.

(5) The examination is 'completely closed book': No books, notes, calculators, PDAs, cell phones, etc.

NOTATIONS USED IN THIS EXAM

- The symbol $\mathbb{N}$ denotes the set of all positive integers
- If n is in $\mathbb{N}$ then $\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers; in particular, $\mathbb{R} = \mathbb{R}^1$ denotes the set of all real numbers.
- Elements of $\mathbb{R}^n$ are sometimes denoted by boldface letters; for example, $\mathbf{u} = (u_1, \ldots u_n)$, where the real numbers $u_1, \ldots \ u_n$ are the *components* of the vector $\mathbf{u}$. Sometimes the vector $\mathbf{u} = (u_1, \ldots u_n)$ is thought of as a column vector.
- If $\mathbf{u} = (u_1, \ldots u_n)$ is an element of $\mathbb{R}^n$, then the **Euclidean norm** or **length** of $\mathbf{u}$ is the number $||\mathbf{u}||_2 = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$.
- $\mathbb{R}^{n\times n}$ denotes sets of all $n \times n$ matrices over $\mathbb{R}$.

Scores	1	2	3	4	5	6	7	8	9	Total

Problem (1) (10 points) Prove or Disprove: There is a continuous real-valued function f on the open unit ball B in $\mathbb{R}^n$ so that the image $f(B) = \mathbb{N}$

Score in # 1

Problem (2) (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and assume that

$$f(\mathbf{0}) = 0 \text{ and } \nabla f(\mathbf{0}) = 0.$$

Consider the following series

$$\sum_{k=1}^{\infty} f\Big(\frac{1}{k}\mathbf{x}\Big).$$

(a) Prove the series converges uniformly on any bounded set in $\mathbb{R}^n$;
(b) Determine if the series is uniformly convergent in $\mathbb{R}^n$ (if yes, prove it, if no, provide a counterexample.)

Score in # 2

Problem (3) (10 points) Consider the matrix-valued function

$$f(M) = M^3, \quad M \in \mathbb{R}^{n\times n}.$$

Is this function differentiable and, if yes, what is its derivative? Justify your answer.

Score in # 3

Problem (4) (10 points)

(a) State the Contraction Mapping Theorem (Banach Fixed Point Theorem) for maps of a complete metric space into itself.

(b) Prove the theorem you stated in Part (a).

Score in # 4

Problem (5) (10 points) Assume $\{a_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of positive numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges.

Score in # 5

Problem (6) (10 points) (a) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at each point of $\mathbb{R}^2$, but f is not continuous on $\mathbb{R}^2$.

(b) Assume that U is an open in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a function so that first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded on U. Prove f is continuous at each point of U.

Score in # 6

Problem (7) (10 points) (a) State the Implicit Function Theorem from $\mathbb{R}^{n+m} \to \mathbb{R}^n$.

(b) Show that the system

$$\begin{cases} x^2 + y^2 + e^u + ye^v = 1 \\ u^2 - v^2 + y + e^{xy} = 0 \end{cases}$$

defines function $u = u(x, y)$ and $v = v(x, y)$ is a neighborhood of $(x, y) = (0, 0)$ so that $(x, y, u(x, y), v(x, y))$ is a solution of the system with $u(0, 0) = 0$ and $v(0, 0) = 1$.

(c) Compute gradient ∇v at $(x, y) = (0, 0)$.

Score in # 7

Problem (8) (10 points) Apply the Divergence Theorem in $\mathbb{R}^n$ to evaluate the following integral:

$$\int_E \frac{y^2}{\sqrt{x^2+4y^2+4z^2}}\,d\sigma$$

where $E = \{(x,y,z) \in \mathbb{R}^3 : 2^{-1}x^2 + y^2 + z^2 = 1\}$ is an ellipsoid in $\mathbb{R}^3$ and $d\sigma$ is the area element on E.

Score in # 8

Problem (9) (10 points) Let $f(x)$ be Riemann integrable on $[0, 2\pi]$ and let

$$g(t) = \int_0^{2\pi} f(x)\sin(tx)dx, \quad t \in \mathbb{R}.$$

(a) Prove $g(t)$ is uniformly continuous on $\mathbb{R}$

(b) Prove $\lim_{n\to\infty} g(n) = 0$.

Score in # 9