Print Your Name: ______________ _______________
last first

Print Your I.D. Number: __________ ______________

Qualifying Examination for Complex Analysis
September 15, 2011
9:00am–11:300am
Room: RH 114

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Notations. Let $D(a, r)$ be the disc in the complex plane $\mathbb{C}$ with center at a and radius r; and $\partial D(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$.

1. Describe all entire holomorphic functions f and g such that

   (a) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^2}$ for all positive integers n. Show your work.

   (b) $g(\frac{1}{n}) = g(-\frac{1}{n}) = \frac{1}{n^3}$ for all positive integers n. Show your work.

2. Let $f(z)$ be an entire holomorphic function such that

$$\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$$

Prove that f is a constant.

3. Evaluate

$$\int_0^\infty \frac{dx}{x^{1/3}(1+x)}.$$

4. (a) Show that there is an analytic function defined in $\Omega = \{z \in \mathbb{C} \mid |z| > 4\}$ whose derivative is

$$f(z) = \frac{z}{(z-1)(z-2)(z-3)}.$$

(b) Does there exist an analytic function in Ω with the derivative

$$g(z) = \frac{z^2}{(z-1)(z-2)(z-3)}\ ?$$

5. Let $f : [0,1] \to \mathbb{C}$ be a continuous function. Define the function $F : \mathbb{C}\backslash[0,1] \to \mathbb{C}$ by

$$F(z) = \int_0^1 \frac{f(t)}{t-z} dt, \quad z \in \mathbb{C}\backslash[0,1].$$

Prove that F is holomorphic on $\mathbb{C}\backslash[0,1]$.

6. Prove the Schwarz-Pick lemma: Let $f : D(0,1) \to D(0,1)$ be holomorphic. Then

$$\left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le \left|\frac{z-a}{1-\bar{a}z}\right|, \quad a, z \in D(0,1).$$

7. Let f be holomorphic in $D(0,1)$ and let

$$M(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

Prove that $M(r,f)$ is an increasing convex function of r on $[0,1)$.

8. Let f be meromorphic in $D(0,1) \setminus \{0\}$ such that

$$\int_{D(0,1)\setminus\{0\}} |f(z)|^3 dA(z) \leq 1.$$

Prove $z = 0$ is a removable singularity of f.