COMPLEX ANALYSIS

Qualifying Exam

Tuesday, September 16, 2010 — 1:00pm - 3:30pm, Rowland Hall 114

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Student's name:

Problem 1.

Let $z_1, \ldots, z_n$ be distinct complex numbers contained in the disk $|z| < R$. Let f be analytic in the closed disk $|z| \leq R$. Let $Q(z) = (z - z_1) \ldots (z - z_n)$. Prove that

$$P(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\zeta|=R} f(\zeta) \frac{1 - \frac{Q(z)}{Q(\zeta)}}{(\zeta - z)} \, \mathrm{d}\zeta$$

is a polynomial of degree $n-1$ having the same values as f at the points $z_1, \ldots, z_n$.

Problem 2.

Show that $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ is meromorphic function on $\mathbb{C}$.

Problem 3.

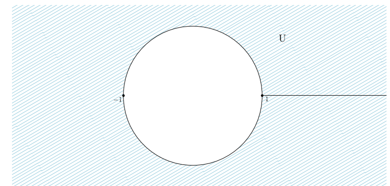
Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc so that for any $f \in \mathcal{F}$, one has

$$\int_D |f(z)|(1-|z|)^2 dA(z) \leq 1.$$

Prove $\mathcal{F}$ is a normal family.

Problem 4.

Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \backslash [1, +\infty)$ to the unit disc.



Problem 5.

Find the integral (where $a > b > 0$)

$$\int_0^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx$$

Problem 6.

Let U be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$, and $z_0 \in U$. Write $f = u + iv$, i.e. u, v are the real and imaginary parts of f. We say that f is complex differentiable at z_0 if $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

(i) Prove that if f is complex differentiable at z_0, then u, v satisfy the Cauchy-Riemann equations.

(ii) Prove that if f is complex differentiable and $f'(z) \neq 0$ in U then f is an orientation preserving conformal map, i.e. for any two differentiable curves α, β in U with $\alpha(0) = \beta(0)$ the angle from $\alpha'(0)$ to $\beta'(0)$ is equal to the angle from $(f \circ \alpha)'(0)$ to $(f \circ \beta)'(0)$.

Problem 7.

(i) State the Mean Value Theorem for analytic functions and use the Cauchy integral formula to prove it.

(ii) Prove that if $f = u + iv$ is an analytic function from an open subset U of $\mathbb{C}$ then the real and imaginary parts u and v of f are harmonic, i.e., $\Delta u = \Delta v = 0$.

(iii) Let U be an open subset of $\mathbb{R}^2$, and $u : U \to \mathbb{R}$ a harmonic function. Prove that if there is $p_0 \in U$ such that $u(p_0) = \inf_{x \in U} u(x)$, then u is a constant.

Problem 8.

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 64$. Determine the region of convergence of the Laurent series

$$\sum_{n=-\infty}^{-1} a_{2|n|} z^{3n} + \sum_{n=0}^{\infty} a_{3n} z^{2n}.$$