Qualifying Exam
Tuesday, September 16, 2010 — 1:00pm - 3:30pm, Rowland Hall 114

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Student’s name:
Problem 1.

Let \( z_1, \ldots, z_n \) be distinct complex numbers contained in the disk \( |z| < R \). Let \( f \) be analytic in the closed disk \( |z| \leq R \). Let \( Q(z) = (z - z_1) \ldots (z - z_n) \). Prove that

\[
P(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} f(\zeta) \frac{1 - \frac{Q(z)}{Q(\zeta)}}{(\zeta - z)} \, d\zeta
\]

is a polynomial of degree \( n - 1 \) having the same values as \( f \) at the points \( z_1, \ldots, z_n \).
Problem 2.

Show that $\sum_{n=1}^{\infty} \frac{1}{z^2 + n\pi}$ is meromorphic function on $\mathbb{C}$. 
Problem 3.

Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc so that for any $f \in \mathcal{F}$, one has
\[
\int_{D} |f(z)|(1 - |z|)^2 dA(z) \leq 1.
\]

Prove $\mathcal{F}$ is a normal family.
Problem 4.

Find an explicit conformal transformation of an open set $U = \{ |z| > 1 \} \setminus [1, +\infty)$ to the unit disc.
Problem 5.

Find the integral (where $a > b > 0$)

$$\int_0^\infty \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} \, dx$$
Problem 6.
Let $U$ be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$, and $z_0 \in U$. Write $f = u + iv$, i.e. $u, v$ are the real and imaginary parts of $f$. We say that $f$ is complex differentiable at $z_0$ if $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

(i) Prove that if $f$ is complex differentiable at $z_0$, then $u, v$ satisfy the Cauchy-Riemann equations.

(ii) Prove that if $f$ is complex differentiable and $f'(z) \neq 0$ in $U$ then $f$ is an orientation preserving conformal map, i.e. for any two differentiable curves $\alpha, \beta$ in $U$ with $\alpha(0) = \beta(0)$ the angle from $\alpha'(0)$ to $\beta'(0)$ is equal to the angle from $(f \circ \alpha)'(0)$ to $(f \circ \beta)'(0)$. 
Problem 7.

(i) State the Mean Value Theorem for analytic functions and use the Cauchy integral formula to prove it.

(ii) Prove that if \( f = u + iv \) is an analytic function from an open subset \( U \) of \( \mathbb{C} \) then the real and imaginary parts \( u \) and \( v \) of \( f \) are harmonic, i.e., \( \Delta u = \Delta v = 0 \).

(iii) Let \( U \) be an open subset of \( \mathbb{R}^2 \), and \( u : U \to \mathbb{R} \) a harmonic function. Prove that if there is \( p_0 \in U \) such that \( u(p_0) = \inf_{x \in U} u(x) \), then \( u \) is a constant.
Problem 8.

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 64$. Determine the region of convergence of the Laurent series

$$\sum_{n=-\infty}^{-1} a_{2|n|} z^{3n} + \sum_{n=0}^{\infty} a_{3n} z^{2n}.$$