Comprehensive Exam in Algebra June 2007 **NAME**__________________________________

PLEASE TRY ALL 10 PROBLEMS.

G1. Show that the alternating group A_4 violates the converse of Lagrange's theorem. (Recall that any subgroup of index 2 is normal.)

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G2. Prove that $11^{48} - 1$ is divisible by 65.

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G3. Prove that an abelian group A with only finitely many subgroups must be finite. *Hint*: First show that every element of A has finite order.

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R1. Let R be an integral domain containing a field F. Then R is a vector space over F. Show that if the dimension of R over F is finite, then R must be a field.

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R2. Let $\mathcal{C} = \{P_u \mid u \in U\}$ be a chain (totally ordered family) of prime ideals in a ring R. Prove that the intersection $I = \bigcap \mathcal{C}$ is a prime ideal. *Hint*: Assume that $ab \in I$ but that $a \notin P_k$ for some $k \in U$. Show that b is in every P_u.

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F1. Let $F < E$ be a field extension. If $a, b \in E$ are algebraic over F, show that $a + b$ is also algebraic over F.

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F2. The complex roots of the binomial $x^n - 1$ ($n \geq 1$) over the rational field $\mathbb{Q}$ are referred to as the **complex nth roots of unity**. Prove the following. ATTENTION: It is NOT acceptable to use the complex form $e^{2\pi ki/n} = \cos(2\pi k/n) + i\sin(2\pi k/n)$ of the roots of unity to answer the questions below. You MUST answer using the algebraic techniques of the class, that is, using the fact that the complex roots of unity are roots over the field $\mathbb{C}$ of the polynomial $x^n - 1 \in \mathbb{Q}[x]$.

a) Prove that there are n *distinct* nth roots of unity.

b) Prove that the set U_n of complex nth roots of unity is a cyclic group under multiplication.

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L1. Let V be a finite-dimensional complex inner product space and let τ be a normal operator on V.

a) Prove that if τ is nilpotent ($\tau^n = 0$ for some $n \geq 1$), then $\tau = 0$.

b) Prove that if $\tau^2 = \tau^3$, then τ is idempotent ($\tau^2 = \tau$).

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L2. Find all 3×3 complex matrices, up to similarity, that satisfy the equation $X^3 = X$.

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L3. Let V be a finite-dimensional inner product space and let T be a linear operator on V. Show that

$$\ker(T^*) = (\operatorname{im}(T))^{\perp}$$

where T^* is the adjoint of T.