Comprehensive Exam in Algebra June 2008 **NAME**__________________________________

PLEASE TRY ALL 10 PROBLEMS.

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G1. Let G be a group and let $G = H \bowtie K$ (where $\bowtie$ is the internal direct sum). Show that if G has the ascending chain condition on normal subgroups, then so does H. Be careful to prove all claims that you make.

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G2. A subgroup H of a group G is **characteristic** if H is σ-invariant for all automorphisms σ of G. Prove that every subgroup of a cyclic group is characteristic.

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G3. Show that no group of order $56 = 7 \cdot 2^3$ is simple. *Hint*: count elements.

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R4. Prove that a finite integral domain R is a field.

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R5. Prove that in an integral domain R, a prime element is irreducible.

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F6. a) If $F < E$ is a field extension of finite degree, prove that E is algebraic over F.

b) Let $F < E$ be a field extension. If $a, b \in E$ are algebraic over F, show that $a + b$ is also algebraic over F.

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F7. Is the polynomial $x^6 - 30x^5 + 6x^4 - 18x^3 + 12x^2 - 6x + 12$ irreducible over the rationals? Explain. (No credit for just yes/no)

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L8. Let V be a finite-dimensional vector space and let T be a linear operator on V. Prove that T is injective if and only if it is surjective.

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L9. Let X be a 3×3 complex matrix. Find all solutions of the equation $X^2 - X = 0$, up to similarity. Use the Jordan canonical form.

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L10. Prove that similar matrices A and B have the same minimal polynomial.