Comprehensive Exam in Analysis

10–1 PM, June 18, 2007

PRINT NAME: ______________________________

- No books
- 9 problems
- 10 points for each problem
- *Show work*
- Good Luck!

SCORE:

1	______
2	______
3	______
4	______
5	______
6	______
7	______
8	______
9	______
Total	______

1. Suppose $a_n > 0$, and $\sum a_n$ diverges. Show that $\sum \frac{a_n}{1+a_n}$ diverges.

2. Let (a, b) be a nonempty open set in $\mathbb{R}$, and f be a function on (a, b). Show the following two definitions are equivalent:

(a) Let $x_0 \in (a, b)$, f is continuous at x_0 iff for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$, $|f(y) - f(x_0)| < \epsilon$.

(b) Let $x_0 \in (a, b)$, f is continuous at x_0 iff for any sequence $\{y_n\}_{n=1}^{\infty} \subset (a, b)$ satisfying $\lim_{n\to\infty} y_n = x_0$, $\lim_{n\to\infty} f(y_n) = f(x_0)$.

3. (a) Carefully <u>state</u> what it means for a sequence $(f_n)_{n\geq 1}$ of real-valued functions defined on an interval I of $\mathbb{R}$ to **converge uniformly on** I.
(b) Prove or Disprove: If $(f_n)_{n\geq 1}$ is a sequence of real-valued functions defined on a metric space X, and if this sequence converges uniformly on X, then the sequence $(g_n)_{n\geq 1}$, defined by $g_n(x) = \arctan(f_n(x))$, also converges uniformly on X.

4. Let X be a metric space. Prove or disprove:

(a) The intersection of finitely many dense subsets of X is dense in X.

(b) The intersection of finitely many open dense subsets of X is open and dense in X.

5. Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ be defined through

$$f(A) = e^{A^2}.$$

where A is a $n \times n$ matrix. Show that f is differentiable and compute its derivative.

6. Let f be a continuous function on $[0,1]$, and

$$S_n = \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}), \qquad n = 1,2,3,\ldots$$

Show that

(a) $\{S_n\}$ is a convergent sequence.

(b) $\lim_{n\to\infty} S_n > 0$, if $f(x) \geq 0$ for all $x \in [0,1]$, and $f(x_0) > 0$ for some $x_0 \in [0,1]$.

7. Suppose that f is continuous for $x \geq 0$, $f(0) = 0$, $f'(x)$ exists and is monotonically increasing for $x \geq 0$. Show that $g(x) = \frac{f(x)}{x}, x > 0$ is monotonically increasing.

8. Consider cubic polynomials of the form $f(x) = x^3 + ax^2 + bx + c$, where a, b and c are real quantities. Note that when $a = 0$, $b = -1$ and $c = 0$ the equation $f(x) = 0$ has three distinct real solutions, namely $u = 1$, $v = -1$ and $w = 0$. Use the Inverse Function Theorem to show that when the coefficients (a, b, c) are sufficiently near $(0, -1, 0)$ then the solutions u, v, w of the equation $x^3 + ax^2 + bx + c = 0$ can be expressed as continuously differentiable functions of the coefficients a, b, c.

9. Let X be a metric space. A function $f : X \to \mathbb{R}$ is called *lower semi-continuous* if

$$f^{-1}((\alpha, \infty)) \text{ is open for any } \alpha \in \mathbb{R}\,.$$

Show that

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0)$$

whenever x_n is a sequence in X with $\lim_{n\to\infty} x_n = x_0$ if f is lower semi-continuous.