Comprehensive Exam in Algebra -- June 2005

Do 10 problems, with AT LEAST 2 PROBLEMS FROM EACH SECTION. That is, you must do

1) At least two problems from the Groups section: G1, G2, G3 and G4.
2) Both problems from the Rings section: R1 and R2.
3) At least two problems from the Fields section: F1, F2 and F3.
4) At least two problems from the Linear Algebra section: L1, L2, L3 and L4.

Do each problem on the sheet for that problem. Write your name on each page. Show all details and quote properly any theorems that you use. All problems are worth 10 points. We prefer complete solutions of a few problems to many partial solutions.

G1. Prove that all groups of order 12 are solvable. Are they all abelian?

G2. a) (4 points) Classify all finite groups with exactly 2 conjugacy classes.
b) (6 points) Classify all finite groups with exactly 3 conjugacy classes.

G3. Is there an example of a group G whose subgroups are all normal but G is not abelian? If not, prove it. If so, describe one such group.

G4. Prove that $7^{120} - 1$ is divisible by 143.

R1. Prove that a finite integral domain is a field.

R2. a) (5 points) If R is an integral domain, show that any prime element is irreducible.
b) (5 points) If R is a unique factorization domain, show that any irreducible element is prime.
(A nonzero element $p \in R$ is **prime** if p is not a unit and if $p \mid ab$ implies $p \mid a$ or $p \mid b$.)

F1. If $F < E$ is a field extension of finite degree, prove that E is algebraic over F.

F2. Let $F < E < \overline{F}$ be field extensions, where $\overline{F}$ is an algebraic closure of F and suppose that E is a splitting field for a family $\mathcal{F}$ of polynomials over F. Prove that any embedding σ of E into $\overline{F}$ over F is an automorphism of E.

F3. Let $F < E$ be a finite Galois extension, with Galois group G. Suppose that $o(G) = 2p^n$ where p is an odd prime and $n > 0$. Prove that there is an intermediate field $F < K < E$ with $K \neq E$ such that $F < K$ is normal.

L1. Let S and T be subspaces of a vector space V. Prove that if $S \cup T$ is a subspace of V then either $S \subseteq T$ or $T \subseteq S$.

L2. Let A be a 4×4 real matrix satisfying the equation $A^4 = 0$. Describe all such matrices A.

L3. Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices over a field F. Consider the linear operator $L(A) = A - A^t$, where A^t is the transpose of A.
a) (5 points) Find the kernel of L and a basis for the kernel.
b) (5 points) Find the image of L and a basis for the image.

L4. Let $\mathbb{C}$ be the complex field. Let S_4 be the permutation group on $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$, where $\mathcal{B}$ is the standard basis for $\mathbb{C}^4$. (For example, $e_1 = (1,0,0,0)$.)
a) (2 points) Each $\sigma \in S_4$ defines a linear operator on $\mathbb{C}^4$. Describe its matrix with respect to $\mathcal{B}$. Use this description to define a homomorphism $\rho: S_4 \to GL(4, \mathbb{C})$, the general linear group of nonsingular 4×4 matrices over $\mathbb{C}$.
b) (2 points) Find the kernel of this homomorphism ρ.
c) (3 points) Show that there is a common eigenvector x for the elements of $\rho(S_4)$.
d) (3 points) Show that the orthogonal complement (under the standard inner product on $\mathbb{C}^4$) of the common eigenvector x is invariant under each member of $\rho(S_4)$.